Electromagnetic Structure of the Nucleon*†

P. FEDERBUSH,‡ M. L. GOLDBERGER, AND S. B. TREIMAN
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
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The electromagnetic structure of the nucleon is studied by using dispersion relation techniques. Contributions to the magnetic moments and mean square radii from the two-pion intermediate state are studied exhaustively. It is shown that the electromagnetic structure of the meson itself may play an important role here; this structure is also discussed. The two-pion state seems to account reasonably for the isotopic vector magnetic moment and magnetization mean square radius, but the charge-density radius appears to be much smaller than the currently accepted experimental value. As regards the isotopic scalar properties of the nucleon, we have studied the contributions from intermediate states with two K mesons and nucleon-antinucleon pairs (more generally baryon pairs). The K-meson state is treated by perturbation theory and found to have a small effect. By use of an argument based on the unitarity of the S matrix, it is shown that the pair contributions must be small. Certain general properties of the three-pion state, believed to be the most important contributor to isotopic scalar quantities, are discussed; but we are unable to make any quantitative statements.

I. INTRODUCTION

QUANTITATIVE experiments on the scattering of electrons by protons and deuterons, carried out by Hofstadter and his collaborators,[1] have provided considerable information on the electromagnetic structure of the nucleon. There have been many theoretical attempts to treat this problem in a semiquantitative way.[2] We mention in particular the recent investigations based on cutoff meson theory,[3] and the relativistic generalizations discussed by Okubo[4] and Tanaka.[5] In the latter approach one attempts to relate the contributions from meson and nucleon currents to the electromagnetic vertex function in terms of scattering amplitudes for pion-nucleon and nucleon-nucleon scattering. Unfortunately, these amplitudes are required for particles "off the mass shell" (i.e., $p^2\neq -m^2$), and the connection with the true physical amplitudes is not known. The precise nature of the approximations which are made (where a definite extrapolation procedure is adopted) is very difficult to assess.

An approach essentially identical in spirit to ours but somewhat less ambitious in scope has been made by Chew, Karplus, Gasiorowicz, and Zachariasen.[6] They have applied dispersion relation methods to the problem and have shown that certain aspects of nucleon structure can be understood from this viewpoint. Since our methods are so similar we shall not describe the work of Chew *et al.* further at this point but shall refer to it at the appropriate places below. A summary of our general procedure and a statement of some of the results have been given in a recent paper.[7] Before entering into detailed calculations, we shall first state the theoretical problem more precisely and give some of the experimental results.

The quantity of most direct theoretical interest is the matrix element of the current density operator j_μ taken between one-nucleon states. This matrix element is related to the vertex operator Γ_μ according to

$$\langle p'|j_\mu|p\rangle = i(m^2/p_0'p_0)^{\frac{1}{2}}q^2\Delta_{Fc}(q^2)\bar{u}(p')\Gamma_\mu(p',p)u(p), \quad (1.1)$$

where $q^2=(p'-p)^2$ is the invariant momentum transfer and Δ_{Fc} is the exact Feynman photon propagation function. The Dirac spinors are normalized according to $\bar{u}(p)u(p)=\pm 1$, for positive- and negative-energy spinors, respectively. To lowest order in the electric charge, $q^2\Delta_{Fc}(q^2)=1$; note that a breakdown of electrodynamics (i.e., a real modification of Δ_{Fc}) would multiply into the structure proper as expressed by $\bar{u}(p')\Gamma_\mu(p',p)u(p)$.

It is both conventional and convenient to express (1.1) in terms of certain scalar functions of q^2. Two equivalent forms which we shall use are

$$\begin{aligned}\langle p'|j_\mu|p\rangle &= (m^2/p_0'p_0)^{\frac{1}{2}}\bar{u}(p')\\ &\quad\times[F_1(q^2)i\gamma_\mu - F_2(q^2)i\sigma_{\mu\nu}(p'-p)_\nu]u(p) \quad (1.2)\\ &= (m^2/p_0'p_0)^{\frac{1}{2}}\bar{u}(p')\\ &\quad\times[G_1(q^2)i\gamma_\mu - G_2(q^2)(p'+p)_\mu]u(p) \quad (1.3)\end{aligned}$$

where $G_1=F_1+2mF_2$ and $G_2=F_2$. That the above structure is the most general one follows from Lorentz and gauge invariance. The function F (G) may be further subdivided into isotopic scalar and vector components according to

$$\begin{aligned}F_1&=F_1{}^S+\tau_3F_1{}^V,\\ F_2&=F_2{}^S+\tau_3F_2{}^V,\end{aligned} \quad (1.4)$$

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‡ National Science Foundation Predoctoral Fellow. Now at Massachusetts Institute of Technology.

[1] R. Hofstadter, Revs. Modern Phys. (to be published).

[2] For earlier references, see B. D. Fried, Phys. Rev. **88**, 1142 (1952).

[3] H. Miyazawa, Phys. Rev. **101**, 1564 (1956), R. Sachs and S. Treiman, Phys. Rev. **103**, 435 (1956), and G. Salzman, Phys. Rev. **105**, 1076 (1957).

[4] S. Okubo, Nuovo cimento **6**, 542 (1957).

[5] K. Tanaka, Phys. Rev. **109**, 578 (1958).

[6] Chew, Karplus, Gasiorowicz, and Zachariasen, Phys. Rev. **110**, 265 (1958), hereafter referred to as C.

[7] J. Bernstein and M. L. Goldberger, Revs. Modern Phys. (to be published).

so that the problem is characterized by four scalar functions. One generally refers to F_1 as the charge density form factor and to F_2 as the magnetization density form factor. The things which are surely known about them are their values at zero momentum transfer:

$$F_1{}^S(0)=F_1{}^V(0)=e/2,\quad F_2{}^S(0)=(\mu_p+\mu_n)/2,\quad F_2{}^V(0)=(\mu_p-\mu_n)/2, \tag{1.5}$$

where e is the proton charge and μ_p and μ_n are the static anomalous magnetic moments of proton and neutron, respectively.

Other experimental knowledge concerning the form factors is rather less certain. If one thinks of the form factors as Fourier transforms of spatial distributions, than $F'(0)$, the derivative evaluated at $q^2=0$, is related to the mean square radius of the spatial distribution. Except for the neutron charge radius, one defines

$$\langle r^2\rangle/6=-F'(0)/F(0). \tag{1.6}$$

Experiments on electron-proton scattering at small q^2 are now being carried out and one will soon have an unambiguous measurement of $\langle r_1{}^2\rangle$ for the proton (this is defined according to (1.6) with $F_1{}^p=F_1{}^S+F_1{}^V$). Preliminary results[1] indicate that $0.18/\mu^2<\langle(r_1{}^2)\rangle<0.32/\mu^2$, where $1/\mu$ is the meson Compton wavelength. The corresponding quantity for the magnetization density is not easy to measure directly since the scattering at small q^2 is dominated by F_1. If one extrapolates with simple functions from the large momentum transfer data, one finds that $\langle r_2{}^2\rangle$ for the proton is in the neighborhood of $0.32/\mu^2$. Needless to say, this is an uncertain and conceivably misleading procedure. Experiments on high-energy electron-deuteron scattering indicate that F_2 for the neutron (at least for the large values of q^2 at which the experiments are carried out) is about the same as for the proton.

The mean square radius for the neutron charge density distribution, conventionally defined as

$$\langle(r_1{}^2)_n\rangle/6=-F_1{}'^n(0)/e, \tag{1.7}$$

with $F_1{}^n=F_1{}^S-F_1{}^V$, is the quantity measured in low-energy electron-neutron scattering. One finds experimentally a very small upper limit on $\langle(r_1{}^2)_n\rangle$, much below the proton value.[8] This constitutes one of the most puzzling features of the whole problem of nucleon structure. We shall define the isotopic scalar and vector radii according to

$$\langle(r_1{}^2)_p\rangle=\tfrac{1}{2}[\langle(r_1{}^2)_S\rangle+\langle(r_1{}^2)_V\rangle], \tag{1.8}$$

$$\langle(r_1{}^2)_n\rangle=\tfrac{1}{2}[\langle(r_1{}^2)_S\rangle-\langle(r_1{}^2)_V\rangle]. \tag{1.9}$$

The smallness of the neutron charge radius implies $\langle(r_1{}^2)_S\rangle\approx\langle(r_1{}^2)_V\rangle$. The mystery is compounded in that, although the scalar charge radius is unexpectedly large, the scalar magnetic moment is small: $\mu_S=(\mu_p+\mu_n)/2\approx-0.06(e/2m)$, as contrasted with $\mu^V=(\mu_p-\mu_n)/2\approx1.8(e/2m)$. While it is not impossible to imagine charge-current distributions with such dual capabilities, they are certainly not very pleasant or simple.

The present discussion of the electromagnetic structure problem is based on dispersion relation techniques. By this we mean that the form factors are represented by expressions such as

$$F_1{}^S(q^2)=\frac{e}{2}-\frac{q}{\pi}\int_{(3\mu)^2}^{\infty}d\sigma^2\frac{\rho_1{}^S(\sigma^2)}{\sigma^2(\sigma^2+q^2-i\epsilon)}, \tag{1.10}$$

$$F_2{}^V(q^2)=\frac{1}{\pi}\int_{(2\mu)^2}^{\infty}d\sigma^2\frac{\rho_2{}^V(\sigma^2)}{\sigma^2+q^2-i\epsilon}, \tag{1.11}$$

where the slight difference in structure and in limits of integration will be explained later. The variable σ^2 represents the square of the mass of the various intermediate states through which the photon-nucleon interaction is effected. The task of the theory is to compute the weight functions ρ which express the contribution of these states.

In Sec. II we discuss the structure of the dispersion relations in detail and describe how the weight functions are to be calculated and what the most important intermediate states are expected to be. In Sec. III the two- and three-pion intermediate states' contributions are treated, as well as the question of the electromagnetic structure of the pion to which one is naturally led. Also discussed in Sec. III is the contribution from intermediate K-meson pairs. The role of nucleon-antinucleon pairs is taken up in Sec. IV. It is shown here—and this is one of the principal results of the present work—that an upper limit on the contribution of such states to the moments and mean square radii can be set by use of the unitarity condition on nucleon-antinucleon scattering. The pairs, in our formulation, can enter in only two angular momentum states and the amplitudes for scattering in these channels are limited in the familiar geometrical way.

Our over-all results concerning nucleon electromagnetic structure can be summarized in the following way. (1) In C it was argued that the isotropic vector moment and radii probably receive their main contributions from intermediate two-pion states and that these contributions can be adequately calculated in perturbation theory. What we find is that the perturbation-theoretic expressions seriously violate unitarity, so that "rescattering" corrections must be significant. We discuss this rescattering but are unable to compute it in any trustworthy way. It is an open question, then, whether or not the isotopic vector properties of the nucleon can be adequately accounted for by the two-pion state. In any case, the apparent agreement between experiment and perturbation theory must be regarded as fortuitous. (2) As for the mystery of the

[8] Hughes, Harvey, Goldberg, and Stafne, Phys. Rev. **90**, 947 (1953).

large isotropic scalar charge radius, our chief result is a negative one. It has often been conjectured that intermediate nucleon-pair states might account for this effect. Our unitarity arguments seem quite reliably to rule this out. The same arguments rule out the possibility of significant contributions from hyperon pairs. We have likewise investigated, this time in perturbation theory, the contributions from intermediate K-meson pairs; it does not seem likely that such states contribute appreciably to the isotopic scalar moment and radii. A theoretically interesting but numerically inaccurate ladder approximation for the pair state is discussed in Appendix A. Among the remaining states of simple configuration, a possible candidate is still the three-pion state. We discuss the general structure of the contributions from this state, but we are completely unable to make any quantitative estimates.

What we achieve here, then, is not a quantitative understanding of the isotopic scalar properties of the nucleon but rather a moderate sharpening of the mystery.

A theoretical process which is very similar to electron-nucleon scattering is scattering of a nucleon by an external mesonic field. Although such fields are rather rare in nature, the process occurs as an intermediate stage in many real reactions. What we are discussing is, of course, the matrix element of the mesonic vertex operator $\Gamma_5(p',p)$, or more exactly, the matrix element of the meson current operator defined as $(\mu^2-\Box)\phi_i=J_i$. The precise connection between the two is

$$\begin{aligned}\langle p'|J_i|p\rangle &= -i(m^2/p_0p_0')^{\frac{1}{2}}(\mu^2+q^2)\Delta_{Fc}\\ &\quad\times(q^2)\bar{u}(p')\Gamma_5(p',p)u(p)\qquad(1.12)\\ &= -i(m^2/p_0p_0')^{\frac{1}{2}}g\bar{u}(p')\tau_i\gamma_5u(p)K(q^2),\end{aligned}$$

where if K is normalized such that $K(-\mu^2)=1$, g is the renormalized Lepore-Watson coupling constant,[9] and Δ_{Fc} is the complete meson propagation function. The quantity $K(q^2)$ satisfies an equation analogous to (1.10) and under certain simplifying assumptions may be calculated quite accurately. These matters are taken up in Appendix B and comparison is made with an early attempt of Edwards[10] to calculate a related quantity. Our result has been used in a discussion of $\pi-\mu$ decay to which the reader is referred for a more heuristic discussion of $K(q^2)$.[11]

Before proceeding to the detailed calculations we would like to say a word about the general theoretical status of the dispersion relations upon which our whole treatment is based. There does not as yet exist a derivation as general and rigorous as the ones which have been given for pion-nucleon scattering. In fact, the only complete derivation that has been given for the electromagnetic or mesonic nucleon vertex function is based on perturbation theory. It is our feeling, however, that the result is correct more generally and we proceed on this basis. In one case, the electromagnetic structure of the meson, a general derivation may be given quite easily. At the appropriate point we will sketch it; for the bulk of the paper we will not concern ourselves with derivations but rather with applications.

II. STRUCTURE OF THE DISPERSION RELATIONS

In this section we shall analyze the general structure of the dispersion relations which we shall encounter. We begin with the electromagnetic nucleon vertex; more precisely we consider a quantity I_μ defined by

$$\begin{aligned}I_\mu &\equiv (p_0p_0'/m^2)^{\frac{1}{2}}\langle p'|j_\mu|p\rangle\\ &= \bar{u}(p')[F_1i\gamma_\mu-F_2i\sigma_{\mu\nu}(p'-p)_\nu]u(p).\qquad(2.1)\end{aligned}$$

We express this in the standard way as[12]

$$I_\mu=\left(\frac{p_0}{m}\right)^{\frac{1}{2}}i\int d^4x\, e^{-ip'\cdot x}\bar{u}(p')\langle 0|(f(x)j_\mu(0))_+|p\rangle,\qquad(2.2)$$

where $f(x)$ is defined as

$$[\gamma_\mu\partial/\partial x_\mu+m]\psi(x)=f(x),\qquad(2.3)$$

and ψ is the nucleon field operator. In writing Eq. (2.2) we have dropped a term which would in general contribute a constant [or at most a polynomial in $(p'-p)^2$] to F_1. The possible existence of such terms will be taken into account when the precise dispersion relations for the F's are given.

We next remark that in place of the time-ordered product we may write

$$\begin{aligned}I_\mu=\left(\frac{p_0}{m}\right)^{\frac{1}{2}}i\int d^4x\, e^{-ip'\cdot x}\bar{u}(p')\\ \times\langle 0|[j_\mu(0),f(x)]\theta(-x)|p\rangle,\qquad(2.4)\end{aligned}$$

where $\theta(-x)$ is zero for $x_0>0$ and unity for $x_0<0$. It would now be the task of a good derivation to show that the coefficient of $\bar{u}(p')$, considered, say, as a function of p_0' in a coordinate system where $\mathbf{p}=0$, has certain definite analyticity properties. For example, since the integrand in (2.4) vanishes for space-like x_μ (the causality condition states that the commutator vanishes in this circumstance), and also for $x_0>0$, we would be led to expect that our function is analytic in the lower half of the p_0' plane. If this were so, it would then be easy to show that the function may be continued into the upper half plane. The remaining problem would be to state the location of the singularities. According to Nambu's perturbation-theory argument (and also, according to one's physical intuition!) one would find

[9] J. Lepore and K. M. Watson, Phys. Rev. **76**, 1157 (1949).
[10] S. F. Edwards, Phys. Rev. **90**, 284 (1953).
[11] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958).
[12] Lehmann, Symanzik, and Zimmermann, Nuovo cimento **1**, 205 (1955).

that there is a branch line from $-\infty$ to $m-2\mu^2/m$ for the isotopic vector form factors, $m-9\mu^2/2m$ for the isotopic scalar.[13] In terms of the invariant variable $q^2=(p'-p)^2$, the branch line in the q^2 plane runs from $-\infty$ to $-(2\mu)^2$ [or $-(3\mu)^2$].

Assuming these things, we can immediately write down the dispersion relations for the form factors, except for the usual uncertainty about the behavior of the functions at infinity. We shall take the dispersion relations to be as follows:

$$F_1{}^S(q^2)=\frac{e}{2}-\frac{q^2}{\pi}\int_{(3\mu)^2}^{\infty} d\sigma^2 \frac{\mathrm{Im}F_1{}^S(-\sigma^2)}{\sigma^2(\sigma^2+q^2-i\epsilon)}, \quad (2.5\mathrm{a})$$

$$F_1{}^V(q^2)=\frac{e}{2}-\frac{q^2}{\pi}\int_{(2\mu)^2}^{\infty} d\sigma^2 \frac{\mathrm{Im}F_2{}^V(-\sigma^2)}{\sigma^2(\sigma^2+q^2-i\epsilon)}, \quad (2.5\mathrm{b})$$

$$F_2{}^S(q^2)=\frac{1}{\pi}\int_{(3\mu)^2}^{\infty} d\sigma^2 \frac{\mathrm{Im}F_2{}^S(-\sigma^2)}{\sigma^2+q^2-i\epsilon}, \quad (2.5\mathrm{c})$$

$$F_2{}^V(q^2)=\frac{1}{\pi}\int_{(2\mu)^2}^{\infty} d\sigma^2 \frac{\mathrm{Im}F_2{}^V(-\sigma^2)}{\sigma^2+q^2-i\epsilon}. \quad (2.5\mathrm{d})$$

The assumed analyticity properties enable us to relate I_μ to the matrix element of j_μ between the vacuum and a state containing a nucleon-antinucleon pair. Since this will be useful for us in our later work, let us note the precise relationship. For an "in" state,

$$\begin{aligned}J_\mu&=\left(\frac{p_0\bar{p}_0}{m^2}\right)^{\frac{1}{2}}\langle 0|j_\mu|\bar{p},p;\mathrm{in}\rangle\\&=-i\left(\frac{p_0}{m}\right)^{\frac{1}{2}}\int d^4x\exp(i\bar{p}\cdot x)\bar{v}(\bar{p})\langle 0|(f(x)j_\mu(0))_+|p\rangle\\&=-i\left(\frac{p_0}{m}\right)^{\frac{1}{2}}\int d^4x\exp(i\bar{p}\cdot x)\bar{v}(\bar{p})\\&\qquad\times\langle 0|[j_\mu(0),f(x)]\theta(-x)|p\rangle,\end{aligned} \quad (2.6)$$

where $v(\bar{p})$ is the spinor which satisfies $(i\gamma\cdot\bar{p}-m)v(\bar{p})=0$. We are led to the association, assuming that the continuation can be made,

$$J_\mu=-\bar{v}(\bar{p})\{F_1[(p+\bar{p})^2]i\gamma_\mu+F_2[(p+\bar{p})^2]i\sigma_{\mu\nu}(p+\bar{p})_\nu\}u(p). \quad (2.7)$$

We are using the phase convention, to which we must consistently adhere, that $|\bar{p}p\rangle=a_{\bar{p}}{}^\dagger|p\rangle$, where $a_{\bar{p}}{}^\dagger$ is the ("in" field) antiparticle creation operator. For an "out" state, which we write as $|\bar{p}p,\mathrm{out}\rangle$, we replace the above form factors by their complex conjugates, which amounts to changing the sign of ϵ in (2.5 a-d). Thus the variable q^2 approaches the real axis from below for $|\bar{p}p,\mathrm{in}\rangle$ and from above for $|\bar{p}p,\mathrm{out}\rangle$.

In order to calculate the imaginary parts of the form factors we write out the absorptive part of I_μ or J_μ. This is the part which arises from the first term in $\theta(-x)=\frac{1}{2}-\frac{1}{2}(x_0/|x_0|)$. We write this absorptive part as $iA_\mu{}^I$ or $iA_\mu{}^J$. Introducing a sum over a complete set of states $|s\rangle$ and carrying out the integrations over x, we find

$$\begin{aligned}A_\mu{}^I&=\pi(p_0/m)^{\frac{1}{2}}\textstyle\sum_s\bar{u}(p')\\&\qquad\times\langle 0|j_\mu|s\rangle\langle s|f|p\rangle\delta(p_s+p'-p)\\&=\bar{u}(p')\{\mathrm{Im}F_1(q^2)i\gamma_\mu\\&\qquad-\mathrm{Im}F_2(q^2)i\sigma_{\mu\nu}(p'-p)_\nu\}u(p);\end{aligned} \quad (2.8)$$

$$\begin{aligned}A_\mu{}^J&=-\pi(p_0/m)^{\frac{1}{2}}\textstyle\sum_s\bar{v}(\bar{p})\\&\qquad\times\langle 0|j_\mu|s\rangle\langle s|f|p\rangle\delta(p_s-\bar{p}-p)\\&=-\bar{v}(\bar{p})\{\mathrm{Im}F_1(\Delta^2)i\gamma_\mu\\&\qquad+\mathrm{Im}F_2(\Delta^2)i\sigma_{\mu\nu}(p+\bar{p})_\nu\}u(p);\end{aligned} \quad (2.9)$$

with $\Delta^2=(p+\bar{p})^2$. In these expressions the δ-function is to be regarded as a Kronecker δ-function insofar as the spatial components of the momenta are concerned (we are using box normalization). That $\mathrm{Im}F_1$ and $\mathrm{Im}F_2$ are indeed the imaginary parts of F_1 and F_2 may be demonstrated using invariance under inversion of motion. In order to make the reality manifest at all stages of approximation, we shall write the sum over states as half the sum over "in" and "out" states, although for brevity we will not explicitly indicate this.

In writing Eq. (2.8) we have, of course, assumed that the spatial integrations may be carried out without difficulty. Actually one encounters formally rising exponentials if $-p_s{}^2<(2m)^2$. Nevertheless, the instruction from perturbation theory is to evaluate the integrals as indicated. Stated more elegantly, one may write p' as $p'=([p_0'^2-\xi]^{\frac{1}{2}}\hat{n},\, ip_0')$, in the system where $\mathbf{p}=0$. Here $\hat{n}$ is a fixed unit vector and ξ is some negative parameter. Then there is no trouble carrying out the integrals. Now assume that the continuation of ξ to the physical value, m^2, may be carried out.

We turn to the question of what states enter the sums in $A_\mu{}^I$ and $A_\mu{}^J$. Since the operator f lowers the nucleon number by unity, $|s\rangle$ must have nucleon number zero. Furthermore it must have zero strangeness and zero total charge. Thus it may consist of pions, even numbers of K mesons, nucleon-antinucleon pairs or more generally baryon pairs of zero strangeness, etc. The least massive of these states would be a one-pion state, but $\langle 0|j_\mu|\pi_0\rangle$ is zero because of charge conjugation invariance: $j_\mu\to -j_\mu$, $|\pi_0\rangle\to|\pi_0\rangle$. Next we encounter a two-pion state. It is easy to show that only the isotopic vector part of j_μ contributes to $\langle 0|j_\mu|2\pi\rangle$.

The case of n pions may be just as easily discussed as far as isotopic spin is concerned. We introduce the operator $G=\mathcal{C}\exp(i\pi I_2)$ where $\mathcal{C}$ is the operation of charge conjugation and $\exp(i\pi I_2)$ generates a rotation of π about 2 axis in isotopic spin space. (I_2 is the 2-

[13] Y. Nambu, Nuovo cimento 6, 1064 (1957) and to be published.

component of the rotation operator in that space.) Then if we write $j_\mu = S_\mu + V_\mu$ where S_μ is an isotopic scalar and V_μ the third component of an isotopic vector, we have $G j_\mu G^{-1} = -S_\mu + V_\mu$. Furthermore, G induces a sign change in all three components of the meson field, so that $G|n\pi\rangle = (-1)^n |n\pi\rangle$. It follows then that the isotopic scalar contributes for states involving an odd number of pions whereas the isotopic vector part of j_μ involves states with an even number. The lower limits in Eqs. (2.5a), (2.5d) reflect these remarks.

Intermediate states consisting of two K particles (K and $\bar{K}$), of course, contribute to both isotopic scalar and isotopic vector form factors. Continuing in this way our enumeration of contributing intermediate states, we ultimately encounter nucleon-antinucleon states, more generally baryon-pair states. It is interesting to note that from the dispersion standpoint it is quite natural when discussing low momentum transfer (small q^2) to expect the two-pion state to enter on a quite different footing from the nucleon-antinucleon state. This is to be contrasted with the perturbation approach, where in lowest order they would be treated together as a unit.

There are a few more general remarks which should be made before we go on to detailed calculations. Since we know the general structure of the various matrix elements we shall encounter, we may use any convenient coordinate system in which to effect the evaluation of $A_\mu{}^I$ and $A_\mu{}^J$. For most purposes $A_\mu{}^J$ is the more convenient quantity and we shall discuss it in a system where the pair p, $\bar{p}$ is at rest: $\mathbf{p}+\bar{\mathbf{p}}=0$. It follows from gauge invariance that $(p+\bar{p})_\mu J_\mu = 0$, so that $J_4 = 0$ in our system. Evidently, then, the states $|s\rangle$ reached in $\langle 0|\mathbf{j}|s\rangle$ must have angular momentum unity and odd parity. For the two-pion state this means we have only a p state and similarly for the $2K$ state. With three pions we have many more possibilities. We must have a π^+, π^-, π^0 configuration and calling the π^+-π^- relative angular momentum l, and the π^0 angular momentum L, we have $l = L = 1, 3, 5, \cdots$. For the nucleon-antinucleon state, only the ${}^3S_1$ and ${}^3D_1$ configurations are relevant.

The significance of these remarks is as follows: for the two-pion state, aside from $\langle 0|j_\mu|\pi_1\pi_2\rangle$, we have to do with the matrix element $\bar{v}(\bar{p})\langle \pi_1\pi_2|f|p\rangle$, with the condition $p+\bar{p} = \pi_1 + \pi_2$. This is proportional to the amplitude for pair annihilation into two pions which are restricted to be in a p state. Unfortunately, $-(p+\bar{p})^2$ begins in our dispersion integrals at $(2\mu)^2$, and hence this is a slightly unphysical process. We shall describe in the next section a method to evaluate this approximately. When $-(p+\bar{p})^2 > 4m^2$, of course, it becomes a physical amplitude and as such can receive contributions only from ${}^3S_1$ and ${}^3D_1$ states whose intensities are limited geometrically by $\pi/[-\frac{1}{4}(p+\bar{p})^2 - m^2]$. This is used to estimate an upper limit to the contributions of the two-pion state for very high values of $\sigma^2 (> 4m^2)$ in Eqs. (2.5b, d). Evidently the various three-pion amplitudes may be estimated in a similar way once $-(p+\bar{p})^2 > 4m^2$.

For the nucleon-antinucleon intermediate state we encounter, besides $\langle 0|j_\mu|N\bar{N}\rangle$, the factor $\bar{v}(\bar{p})\langle N\bar{N}|f|p\rangle$, with the restriction $N+\bar{N} = p+\bar{p}$. This is directly proportional to the *physical* amplitude for nucleon-antinucleon scattering. Our angular momentum considerations tell us that we need only the amplitudes for ${}^3S_1 \to {}^3S_1$, ${}^3S_1 - {}^3D_1$, and ${}^3D_1 \to {}^3D_1$ (${}^3D_1 \to {}^3S_1$ is the same as ${}^3S_1 \to {}^3D_1$). These are again geometrically limited and this enables us to put an upper limit on the contribution of the pair intermediate state.

We note in passing that in the case of the meson-nucleon vertex $\langle p'|J_i|p\rangle$, we would discuss $\langle 0|J_i|\bar{p}p\rangle$, which again may be treated in the rest system $\mathbf{p}+\bar{\mathbf{p}} = 0$. Since J_i is a pseudoscalar, the state $|\bar{p}p\rangle$ must be ${}^1S_0$. Similarly all of the intermediate states in the expression for the absorptive part similar to that for $A_\mu{}^J$ must have angular momentum zero and odd parity. The first few relevant states are those with 3, 5, $\cdots$ pions, 4, 6, $\cdots$ K particles, $N\bar{N}$, etc.

III. CONTRIBUTIONS FROM MESON STATES

A. Two-Pion State

We begin our detailed discussion by considering the least massive of the states described in Sec. II, namely that consisting of two pions. Although the contributions from this state have been analyzed by Chew *et al.*,[6] for completeness we shall repeat some of their work. In addition we shall take into account certain effects which they did not consider.

In lowest order perturbation theory the contribution from the two-meson state may be described as follows: the nucleon emits a virtual meson which interacts as a point particle with the electromagnetic field and is then reabsorbed by the nucleon. This term may be separated in a gauge-invariant way from its natural partner in perturbation theory, namely, the term in which the nucleon, after emitting a virtual meson, interacts with the electromagnetic field. The two obvious modifications of the perturbation treatment of the meson current contribution which are suggested by our dispersion approach are that the structure of the meson's electromagnetic interaction be taken into account and that the emission and reabsorption of the virtual meson be treated more accurately. Expressed in the terms of (2.9), what we encounter with the two-pion intermediate state are the matrix elements $\langle 0|j_\mu|\pi\pi\rangle$ and $\langle \pi\pi|f|p\rangle$. The former is determined by the electromagnetic structure of the meson; the latter is proportional to the matrix element which describes nucleon pair annihilation into two mesons.

The predictions of perturbation theory, calculated directly by Feynman techniques or by our dispersion formulas given below, are as follows:

$$G_2{}^V(0)=\frac{e}{2m}\left(\frac{g^2}{4\pi}\right)\left(\frac{1}{2\pi}\right)\Big[(1-2\eta)-\eta(2-\eta)\ln\eta-\left(\frac{\eta}{1-\eta/4}\right)^{\frac{1}{2}}(2-4\eta+\eta^2)\cos^{-1}(\eta^{\frac{1}{2}}/2)\Big], \quad (3.1)$$

$$G_2'{}^V(0)=\frac{e}{2m}\left(\frac{g^2}{4\pi}\right)\left(\frac{1}{8\pi m^2}\right)\Big[\frac{4\eta^2-(62/3)\eta+20}{\eta-4}+[-2\eta^2+(16/3)\eta-2]\ln\eta-\frac{2\cos^{-1}(\eta^{\frac{1}{2}}/2)}{(4\eta-\eta^2)^{\frac{1}{2}}}\times\left(\frac{-2\eta^5+(52/3)\eta^4-46\eta^3+36\eta^2-(8/3)\eta}{\eta^2-4\eta}\right)\Big], \quad (3.2)$$

$$G_1'{}^V(0)=e\left(\frac{g^2}{4\pi}\right)\left(\frac{1}{16\pi m^2}\right)\Big[\left(-\frac{10}{3}+\frac{4}{3}\eta\right)+\left(-\frac{2}{3}\eta^2+\frac{8}{3}\eta-2\right)\ln\eta-2\frac{(-\frac{2}{3}\eta^3+4\eta^2-6\eta+\frac{4}{3})}{(4\eta-\eta^2)^{\frac{1}{2}}}\times\cos^{-1}\left(\frac{\eta^{\frac{1}{2}}}{2}\right)\Big], \quad (3.3)$$

where $\eta=(\mu/m)^2=0.022$. The derivatives in (3.2) and (3.3) are taken with respect to the squared momentum transfer $q^2=(p'-p)^2$. Using $g^2/4\pi=15$, we find the following numerical results:

$$\begin{aligned}
G_2{}^V(0)&=\frac{\mu_p-\mu_n}{2}=1.67\left(\frac{e}{2m}\right);\\
G_2'{}^V(0)&=-\frac{1.48}{m^2}\left(\frac{e}{2m}\right);\quad G_1'{}^V(0)=-\frac{2.36}{m^2}e;\\
\langle(r_1{}^2)_V\rangle&=-\frac{12}{e}[G_1'{}^V(0)-2mG_2'{}^V(0)]=\frac{0.24}{\mu^2};\\
\langle(r_2{}^2)_V\rangle&=-6\frac{G_2'{}^V(0)}{G_2{}^V(0)}=\frac{0.12}{\mu^2}.
\end{aligned} \quad (3.4)$$

We see that the predicted vector magnetic moment agrees well with the experimental value $1.86(e/2m)$. If we assume that $\langle(r_1{}^2)_V\rangle\approx\langle(r_1{}^2)_S\rangle$ as would be indicated from the neutron-electron interaction experiments, we would have for the proton $\langle(r_1{}^2)_p\rangle=0.24/\mu^2$, which agrees quite well with the experimental results which put this quantity in the range $0.18/\mu^2$ to $0.32/\mu^2$. Neglecting the isotopic scalar contribution to the magnetization radius, we have $\langle(r_2{}^2)_p\rangle\approx 0.12/\mu^2$, which seems rather small, although as we have commented $\langle(r_2{}^2)_p\rangle$ is not directly measured experimentally.

We turn now to the dispersion relation treatment. For convenience we record again Eq. (2.9), writing in explicitly the two-pion state now under discussion (recall that we eventually take half the sum over "in" and "out" states):

$$A_\mu{}^J(2\pi)=-\pi\left(\frac{p_0}{m}\right)^{\frac{1}{2}}\sum_{ij}\frac{d^3qd^3k}{(2\pi)^3}\bar{v}(\bar{p})\langle 0|j_\mu|q_ik_j\rangle\times\langle q_ik_j|f|p\rangle\delta(q+k-p-\bar{p}), \quad (3.5)$$

where the indices i and j are isotopic labels. From gauge invariance and isotopic spin requirements it follows that the first matrix element in (3.5) may be written

$$(4k_0q_0)^{\frac{1}{2}}\langle 0|j_\mu|q_ik_j\text{ out}\rangle=i(e/\sqrt{2})\epsilon_{3ij}(q-k)_\mu M^*[(q+k)^2], \quad (3.6)$$

where $M^*(0)=1$. (The reason for writing M^* is that it will be our convention to define the various form factors of our theory in terms of the "natural" order for the states; i.e., initial states are "in", final states are "out.") The form factor M will be studied later by dispersion relation methods; for the moment we need only the structure of (3.6). The important thing to notice here is that in the rest frame of the pion pair ($\mathbf{q}+\mathbf{k}=0$), we deal only with states of total angular momentum unity; also in this frame we see that gauge invariance, $(q+k)_\mu\langle 0|j_\mu|q_ik_j\rangle=0$, implies $\langle 0|j_4|q_ik_j\rangle=0$.

Consider next the matrix element $\bar{v}(\bar{p})\langle q_ik_j\text{ out}|f|p\rangle$, where $q+k=p+\bar{p}$. This matrix element describes nucleon pair annihilation into two pions; however, it is required for unphysical values of the total energy of the system. This may be seen by noting the consequences of the δ-function in (3.5). In the rest frame of the meson pair we have the energy condition $4(\mu^2+|\mathbf{q}|^2)=-(p+\bar{p})^2$, where $\mathbf{q}$ is the pion momentum. One sees that in Eq. (2.5) the dispersion variable $-(p+\bar{p})^2$ can be as small as $4\mu^2$ (corresponding to $\mathbf{q}=0$). Once this variable exceeds $4m^2$ we are, of course, in the physical region.

It is easy to see that the matrix element in question may be written

$$\sqrt{2}(4k_0q_0p_0/m)^{\frac{1}{2}}\bar{v}(\bar{p})\langle q_ik_j\text{ out}|f|p\rangle=\bar{v}(\bar{p})\{A_{ij}-i\gamma\cdot[\tfrac{1}{2}(q-k)]B_{ij}\}u(p), \quad (3.7)$$

where

$$(A_{ij},B_{ij})=(A_1,B_1)\delta_{ij}+(A_2,B_2)\tfrac{1}{2}[\tau_i,\tau_j]; \quad (3.8)$$

and the A's and B's are to be regarded as functions of $\Delta^2=(q+k)^2$ and $\nu=-(p-\bar{p})\cdot(q-k)/4m$. Evidently only the charge exchange amplitudes A_2 and B_2 contribute in (3.5). Inserting (3.6)–(3.8) into (3.5), we now have

$$A_\mu{}^J(2\pi)=\frac{\pi e}{4}\int\frac{d^3qd^3k}{(2\pi)^3q_0k_0}(q-k)_\mu\delta(q+k-p-\bar{p})\bar{v}(\bar{p})\times\{\mathrm{Re}(M^*A_2)-i\gamma\cdot[\tfrac{1}{2}(q-k)]\,\mathrm{Re}(M^*B_2)\}\tau_3u(p), \quad (3.9)$$

where we have carried out the isotopic spin summations and taken the symmetrized sum over "in" and "out" states.

We now reduce (3.9) further by going over to the variables ν and Δ^2. Separating out the contributions to $\mathrm{Im}F_1$ and $\mathrm{Im}F_2$ we find

$$\mathrm{Im}F_1{}^V(\Delta^2)=\frac{e}{16\pi}(-\Delta^2)^{-\frac{1}{2}}\,\mathrm{Re}M^*(\Delta^2)\int_{-PQ/m}^{PQ/m}d\nu \times\left[\frac{2m^3}{P^3}\nu A_2(\nu,\Delta^2)-\frac{mQ^3}{PQ}\left\{\frac{m^2}{P^2}\left(\frac{3m^2\nu^2}{P^2Q^2}-1\right)-\left(1-\frac{m^2\nu^2}{P^2Q^2}\right)\right\}B_2(\nu,\Delta^2)\right]\theta(-\Delta^2-4\mu^2), \quad (3.10)$$

$$\mathrm{Im}F_2{}^V(\Delta^2)=\frac{e}{32\pi}(-\Delta^2)^{-\frac{1}{2}}\,\mathrm{Re}M^*(\Delta^2)\int_{-PQ/m}^{PQ/m}d\nu \times\left[\frac{m^2Q^3}{P^3Q}\left(\frac{3m^2\nu^2}{P^2Q^2}-1\right)B_2(\nu,\Delta^2)-\frac{2m^2\nu}{P^3}A_2(\nu,\Delta^2)\right]\theta(-\Delta^2-4\mu^2), \quad (3.11)$$

where

$$P=(\tfrac{1}{4}\Delta^2-m^2)^{\frac{1}{2}};\quad Q=(-\tfrac{1}{4}\Delta^2-m^2)^{\frac{1}{2}};$$

and one is instructed to take the real part of everything to the right of Re.

B. Formulation in Terms of Pion-Nucleon Scattering

To proceed further we must know how the amplitudes A_2 and B_2 depend on the variables ν and Δ^2. As already mentioned, they can be regarded as amplitudes for nucleon pair annihilation into two pions; but we require the continuation of these amplitudes into an unphysical region. We can also regard A_2 and B_2 as continuations of pion-nucleon scattering amplitudes[14]; as in C, this is the approach which we shall follow here. The possibility of making this continuation is based on the observation that $\langle q_ik_j\,\mathrm{out}|f|p\rangle\sim\langle q_i|f|-k_jp\,\mathrm{in}\rangle$, from which it follows that the variable ν may be identified with the analogous quantity defined in the scattering problem; and our variable $\Delta^2=(q+k)^2$ is identical with the momentum transfer variable in the scattering problem. The precise relation is as follows: If the scattering process $l+p\rightarrow q+p'$ is described by the matrix element

$$(4l_0q_0p_0/m)^{\frac{1}{2}}\bar{u}(p')\langle q_i|f|l_jp\,\mathrm{in}\rangle =\bar{u}(p')F(p',q_i;\,p,l_j)u(p), \quad (3.12)$$

[14] For a discussion of pion-nucleon scattering in terms of dispersion relations, see Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1337 (1957).

then our matrix element can be written

$$\sqrt{2}(4k_0q_0p_0/m)^{\frac{1}{2}}\bar{v}(\bar{p})\langle q_ik_j\,\mathrm{out}|f|p\rangle =\bar{v}(\bar{p})F(-\bar{p},\,q_i;\,p,\,-k_j)u(p). \quad (3.13)$$

What we can now do, then, is write down the pion-nucleon dispersion relations for A_2 and B_2:

$$A_2(\nu,\Delta^2)=\frac{1}{\pi}\int_{\mu-\Delta^2/4m}^{\infty}d\nu'\,\mathrm{Im}A_2(\nu',\Delta^2)\times\left\{\frac{1}{\nu'-\nu-i\epsilon}-\frac{1}{\nu'+\nu-i\epsilon}\right\}; \quad (3.14)$$

$$B_2(\nu,\Delta^2)=\frac{g^2}{2m}\left[\frac{1}{\nu-(\mu^2/2m)-(\Delta^2/4m)}-\frac{1}{\nu+(\mu^2/2m)+(\Delta^2/4m)}\right]+\frac{1}{\pi}\int_{\mu-\Delta^2/4m}^{\infty}d\nu'\,\mathrm{Im}B(\nu',\Delta^2)\times\left\{\frac{1}{\nu'-\nu-i\epsilon}+\frac{1}{\nu'+\nu-i\epsilon}\right\}, \quad (3.15)$$

It should be emphasized that in the present problem we are concerned with values of Δ^2 which are negative, a situation not envisaged in the usual derivations of the pion-nucleon dispersion relations. We have, however, verified up to fourth order in perturbation theory that the extension to $\Delta^2=-4m^2$ which we ultimately require (see below) is legitimate and we conjecture that this is a general result.

Even with all this formal manipulation we are still faced with the problem of determining how $\mathrm{Im}A_2$ and $\mathrm{Im}B_2$ depend on their arguments ν and Δ^2 over the range which concerns us. Physical pion-nucleon scattering is characterized by positive Δ^2 and $\nu>(m^2+\Delta^2/4)^{\frac{1}{2}}(\mu^2+\Delta^2/4)^{\frac{1}{2}}/m$. However, in our problem, as already said, Δ^2 is always negative. The only way known at present to deal with this situation is to continue $\mathrm{Im}A_2$ and $\mathrm{Im}B_2$ from the physical region by means of a Legendre polynomial expansion. The legitimacy of such a procedure is not assured. It has been established by Lehmann[15] that one may in fact use this method for Δ^2 up to about $32\mu^2/3$. What the precise situation is for our negative Δ^2 we do not know; but it is our feeling, and this is the point of view adopted in C, that one might get at least qualitative indications by using a finite number of Legendre polynomials. We shall return to this formulation below. For the moment, we shall make a digression and discuss the limitations imposed by the requirement of unitarity.

C. The Unitarity Condition

Once the dispersion variable $-\Delta^2$ exceeds $(2m)^2$, the matrix element $\bar{v}(\bar{p})\langle q_ik_j\,\mathrm{out}|f|p\rangle$ describes physical nucleon pair annihilation into two pions. Here an entirely different approach is possible. We have already

[15] H. Lehmann (to be published).

noted that the matrix element $\langle 0|j_\mu|q_ik_j\rangle$ selects only two-pion states with angular momentum unity. It is evident, then, that for the pair-annihilation matrix element we are concerned only with pairs which are in $^3S_1$ and $^3D_1$ states of isotopic spin unity. The magnitudes of the contributions from each of these channels is limited in the usual way by unitarity and we can therefore set an upper limit on the contributions from the two-pion state for the region $-\Delta^2>(2m)^2$. Of course, the whole contribution of the two-pion state could be formulated *ab initio* from the annihilation standpoint, but the use of unitarity is limited to the physical region.

Let us return to (3.5) and evaluate $A_\mu{}^J(2\pi)$ in the rest frame of the nucleon pair, where, as we have noted, $A_4{}^J=0$. We now reduce all Dirac spinors to Pauli 2-component spinors, χ, in this reference frame. Thus, from (2.9) we have

$$\mathbf{J}=\frac{p_0}{m}\langle 0|\mathbf{j}|p\bar{p}\rangle=-\frac{p_0}{m}\chi_{\bar{p}}{}^*\left[\left\{\frac{2p_0+m}{3p_0}F_1+\frac{2m+p_0}{3m}2mF_2\right\}\boldsymbol{\sigma}+\left\{\frac{p_0-m}{3m}2mF_2-\frac{p_0-m}{3p_0}F_1\right\}\{3\boldsymbol{\sigma}\cdot\hat{p}\hat{p}-\boldsymbol{\sigma}\}\right]\chi_p, \quad (3.16)$$

where $\hat{p}=\mathbf{p}/p$. The structure of (3.16) makes it evident that only $^3S_1$ and $^3D_1$ pair states are involved in this problem.

To evaluate the absorptive part $\mathbf{A}^J(2\pi)$ we could now express the angular momentum decomposition of $\bar{v}(\bar{p})\langle q_ik_j|f|p\rangle$ in terms of the amplitudes A_2 and B_2 introduced earlier. But it is more convenient to simply express this matrix element directly in terms of Pauli spinors and effect the calculation in the rest frame of the nucleon pair. We therefore write

$$\bar{v}(\bar{p})\langle q_ik_j|f|p\rangle=-\pi(\tfrac{3}{2}mq_0|\mathbf{q}|^3|\mathbf{p}|)^{\frac{1}{2}}\chi_{\bar{p}}{}^*\tfrac{1}{2}[\tau_i,\tau_j]\times\{\sqrt{2}\beta_S\boldsymbol{\sigma}\cdot\mathbf{q}-\beta_D[3\boldsymbol{\sigma}\cdot\hat{p}\hat{p}\cdot\mathbf{q}-\boldsymbol{\sigma}\cdot\mathbf{q}]\}\chi_p, \quad (3.17)$$

where the kinematic factors have been so chosen that β_S and β_D are just the S-matrix elements for production of a p-wave pion pair by a nucleon pair in the $^3S_1$ and $^3D_1$ states, respectively. The amplitudes β_S and β_D are to be regarded as functions of $p_0=(-\Delta^2/4)^{\frac{1}{2}}$. For physical processes $(-\Delta^2>4m^2)$ we can apply the unitarity condition, which tells us that $|\beta_S|$, $|\beta_D|\leqslant 1$.

Finally, we evaluate (3.6) in the rest frame of the pions, insert this along with (3.17) into (3.5), evaluate $\mathbf{A}^J(2\pi)$ and thus find the two-pion contribution to $\mathrm{Im}F_1{}^V$ and $\mathrm{Im}F_2{}^V$. We find

$$\mathrm{Im}F_1{}^V(\Delta^2)=e\left(\frac{3|\mathbf{q}|^3}{16|\mathbf{p}|p_0{}^2}\right)^{\frac{1}{2}}\mathrm{Re}M^*\times\left[\frac{p_0}{p_0+m}\left(\frac{\sqrt{2}\beta_S+\beta_D}{3}\right)+\frac{mp_0}{|\mathbf{p}|^2}\beta_D\right], \quad (3.18)$$

$$\mathrm{Im}F_2{}^V(\Delta^2)=\frac{e}{2}\left(\frac{3|\mathbf{q}|^3}{16|\mathbf{p}|p_0{}^2}\right)^{\frac{1}{2}}\mathrm{Re}M^*\times\left[\left(\frac{1}{p_0+m}\right)\left(\frac{\sqrt{2}\beta_S+\beta_D}{3}\right)-\frac{p_0}{|\mathbf{p}|^2}\beta_D\right]. \quad (3.19)$$

We imagine that all quantities are expressed as functions of $\Delta^2=(p+\bar{p})^2$.

In Eqs. (3.10) and (3.11) on the one hand, and (3.18) and (3.19) on the other, we have two alternate formulations of the problem of computing the contributions to $\mathrm{Im}F_1$ and $\mathrm{Im}F_2$ from the two-pion intermediate state. With either formulation, the final computation of the two-pion contribution to the real parts of the form factors is to be effected by use of the dispersion relations (2.5), where the integration variable Δ^2 ranges from $-(2\mu)^2$ to $-\infty$. In the formulation represented by (3.18) and (3.19), the amplitudes β_S and β_D are restricted by unitarity to absolute value less than or equal to unity, provided $-\Delta^2>4m^2$. For $-\Delta^2<4m^2$, β_S and β_D are continuations of the $^3S_1$ and $^3D_1$ annihilation amplitudes and here the unitarity restriction does not apply

D. Quantitative Estimates

Before a complete discussion of the two-pion contribution can be given, it is necessary to make a study of the pion electromagnetic form factor $M(\Delta^2)$. This is a major undertaking in its own right; and in order not to mix up too many effects at one time, we prefer to put this off until later. Here we shall adopt the customary approximation of setting $M(\Delta^2)=1$, which is tantamount to treating the pion as structureless. As we shall later see, this may be a drastic approximation.

It has been argued in C that the effect of the rescattering terms in (3.14) and (3.15) is not significant in the nucleon structure problem; that is, one can set $A_2=0$ and for B_2 retain only the Born term, i.e., neglect the integral in (3.15). If this approximation is adopted, the dispersion relations (2.5) yield precisely the results of lowest order perturbation theory, already discussed in subsection A. What we shall show here is that this approximation, while it yields fair agreement with experiment, is quite unjustified in principle, at least as regards the magnetic moment and charge density radius; i.e., the agreement with experiment must be looked upon as fortuitous.

The point is that in the dispersion integrals (2.5) (where $\sigma^2=-\Delta^2$ is the dispersion variable) the region of integration $\sigma^2>4m^2$ contributes far too much to the charge radius and magnetic moment if one adopts the Born approximation; i.e., unitarity is badly violated. One finds, for example, that the calculated magnetic moment, $1.67(e/2m)$, receives a contribution of $0.8(e/2m)$ from this region of integration. That this is too large one sees by computing the annihilation amplitudes β_S and β_D in perturbation theory. For

$-\Delta^2 > 4m^2$, the resulting expressions exceed the unitarity limit $|\beta_S|$, $|\beta_D| \leqslant 1$ by factors of three or four over the important range of integration in (2.5). We can obtain an upper limit on the contributions in (2.5) coming from the range $\sigma^2 > 4m^2$ by setting the amplitudes β_S and β_D at their maximum values in (3.18) and (3.19). For the magnetic moment we find that the maximum contribution from the region $\sigma^2 > 4m^2$ is only $0.2(e/2m)$; and this is probably far larger than the true contribution. Even if we were to adopt this upper limit and add to it the perturbation theory result for $\sigma^2 < 4m^2$, we would find for the moment the value $(0.87+0.2)(e/2m) = 1.1(e/2m)$. This is to be compared with the experimental value $1.85(e/2m)$. The agreement is no longer so impressive. Moreover, in view of the fact that perturbation theory is so badly in error for $\sigma^2 > 4m^2$, there is no compelling reason to trust it for $\sigma^2 < 4m^2$. In connection with the magnetization density radius, the violation of unitarity is less significant numerically, since this quantity does not receive much contribution from the range $\sigma^2 > 4m^2$, even in perturbation theory.

We are faced with at least three possibilities: (1) The rescattering corrections below $4m^2$ are quite important, just as they must be above $4m^2$; and if they were treated correctly one would find that indeed most of the vector magnetic moment comes from the low σ^2 part of the two-pion configuration. (2) The meson vertex function $\langle 0|j_\mu|\pi\pi\rangle$ must be treated properly before one can get agreement with experiment. (3) The two-pion state alone cannot account for the low momentum transfer properties of nucleon electromagnetic structure and one must look to more massive configurations for unexpectedly large contributions. We shall discuss these possibilities in turn.

Let us first attempt an estimate of the rescattering corrections. We base this treatment on the formulas (3.10), (3.11), (3.14), and (3.15) developed above. As in C, the Δ^2 dependence of $\mathrm{Im}A_2$ and $\mathrm{Im}B_2$ is obtained from a Legendre polynomial expansion, in which we include only the contribution from the (3,3) amplitudes in pion-nucleon scattering. However, we make no nonrelativistic approximation. The relevant formulas are as follows:

$$\frac{A_2(\nu', -\sigma^2)}{4\pi} = 3\frac{W'+m}{E'+m} f_3^{(-)}\left(1+\frac{\sigma^2}{2k'^2}\right) + \frac{W'-m}{E'-m} f_3^{(-)}, \quad (3.20)$$

$$\frac{B_2(\nu', -\sigma^2)}{4\pi} = \frac{3}{E'+m} f_3^{(-)}\left(1+\frac{\sigma^2}{2k'^2}\right) - \frac{1}{E'-m} f_3^{(-)}; \quad (3.21)$$

where k' and E' are, respectively, the momentum and energy of the nucleon in the center-of-mass system (and are to be regarded as functions of ν' and $\sigma^2 = -\Delta^2$); and

$$W'^2 = [(\mu^2+k'^2)^{\frac{1}{2}} + (m^2+k'^2)^{\frac{1}{2}}]^2 = m^2+\mu^2+2m(\nu'-\sigma^2/4m). \quad (3.22)$$

The amplitude $f_3^{(-)}$ is related to the pion nucleon scattering amplitudes in the $J=\frac{3}{2}$, $I=\frac{1}{2}$ and $J=\frac{3}{2}$, $I=\frac{3}{2}$ states according to

$$f_3^{(-)} = \tfrac{1}{3}[f(\tfrac{3}{2},\tfrac{1}{2}) - f(\tfrac{3}{2},\tfrac{3}{2})]. \quad (3.23)$$

For the dependence of $f_3^{(-)}$ on W', the total center-of-mass energy, we make essentially the same approximation as in C, namely,

$$\frac{2m}{\pi k'}\frac{4\pi\,\mathrm{Im}f_3^{(-)}}{k'} = -\frac{4}{9}g^2\left(1+\frac{w_r}{m}\right)\delta[W'^2-(m+w_r)^2], \quad (3.24)$$

where w_r is the energy of the (3,3) resonance ($w_r \approx 2\mu$). We now use these expressions to compute $\mathrm{Im}F_2$ and substitute the result in (2.5). For the magnetic moment the integration interval $(2\mu)^2 < \sigma^2 < (2m)^2$ is found to contribute the value $(0.87+1.03)(e/2m) = (1.90)(e/2m)$. The first term, 0.87, is the Born contribution, whereas the second represents the rescattering. In all probability the latter contribution is overestimated since our Legendre polynomial continuation procedure may be diverging badly for the large negative momentum transfers ($\Delta^2 \sim -4m^2$) of importance here. It is perhaps reassuring that the sign and approximate size of the rescattering contributions restore the fairly good agreement with experiment that had previously been obtained (unjustifiably) with perturbation theory. Notice that the Legendre polynomial continuation was not, and could not be, extended to all $\sigma^2 > 4m^2$: if this were done, the dispersion integrals would diverge badly.

Our feeling is that the two-pion state may well largely account for the vector magnetic moment, in which we concur with C. What we do not agree with is that the quantitative estimate can be meaningfully based on perturbation theory. In C, the assertion that the rescattering corrections are small ($\sim 17\%$) was based on an unwarranted ($1/m$) expansion and an imprecise integration. Our conclusion is that one must effectively disregard the contribution to the dispersion integral from masses greater than $2m$ and for smaller masses must take careful account of rescattering. The rescattering effect is so large, however, that we do not have much confidence in the procedure we used, but the results obtained suggest that a careful treatment might lead to good agreement with experiment. One must, of course, also expect some contributions to the moment from more massive intermediate states.

Thus far our discussion has been confined largely to the magnetic moment. The magnetization density radius is not greatly effected by the rescattering. Taking into account only the region $4\mu^2<\sigma^2<4m^2$, we find that $\langle r_2^2\rangle$ increases from $0.12/\mu^2$, the perturbation value, to $0.16/\mu^2$. It should be noted that in calculating $\langle r_2^2\rangle$ we normalize in each case with respect to the theoretical magnetic moment. The charge density radius, however, changes drastically, going from $0.24/\mu^2$ to $0.033/\mu^2$, in apparently strong disagreement with experiment. The complete numerical situation will be summarized after we have discussed the question of the meson's electromagnetic structure.

E. Meson Electromagnetic Form Factor

The last point which must be discussed in connection with the two-pion state concerns the meson form factor $M(\Delta^2)$, which until now we have set equal to unity in our numerical estimates. We now study this quantity with dispersion relation methods.

Consider the quantity M_μ defined [see Eq. (3.6)] by

$$M_\mu(q_ik_j)=(4q_0k_0)^{\frac{1}{2}}\langle 0|j_\mu|q_ik_j\text{ in}\rangle =i(e/\sqrt{2})\epsilon_{3ij}(q-k)_\mu M[(q+k)^2], \quad (3.25)$$

where, as we recall, the factors are chosen so that $M(0)=1$. As remarked earlier, only the isotopic vector part of j_μ contributes here. In the standard way we find

$$M_\mu=iq_0^{\frac{1}{2}}\int d^4x\, e^{ik\cdot x}(\mu^2-\Box_x) \times\langle 0|(j_\mu(0)\phi_j(x))_+|q^i\rangle. \quad (3.26)$$

When the indicated operations are carried out, there appear terms coming from equal-time commutators; on invariance grounds and from the assumed locality of the theory, such terms must have the form $(q-k)_\mu$ times polynomials in $(q+k)^2$. We shall assume that in fact only a constant times $(q-k)_\mu$ appears; and this constant will later be fixed, through use of a subtracted dispersion relation, to guarantee the condition $M(0)=1$. We therefore drop these terms for the moment and write simply

$$M_\mu=iq_0^{\frac{1}{2}}\int d^4x\, e^{ik\cdot x}\langle 0|(J_j(x),j_\mu(x))_+|q^i\rangle. \quad (3.27)$$

We obtain a final form for M_μ by writing $(j_\mu(0)J_j(x))_+ =[j_\mu(0),J_j(x)]\theta(-x_0)+J_j(x)j_\mu(0)$ and noting that the last term makes no contribution for $k_0>\mu$. Thus

$$M_\mu=iq_0^{\frac{1}{2}}\int d^4x\, e^{ik\cdot x}\langle 0|[j_\mu(0),J_j(x)]\theta(-x_0)|q_i\rangle. \quad (3.28)$$

Now it follows from the dynamical independence of the vector potential and the meson field that $[j_\mu(0),J_j(x)]$ vanishes for space-like x. Thus M_μ is the Fourier transform of a function which vanishes everywhere except in the past light cone. This suggests that it should be possible to derive a dispersion relation for M_μ, or, more properly, for the scalar function $M[(q+k)^2]$. Note that we can isolate the latter in an obvious way to obtain

$$[-4\mu^2-(q+k)^2]M[(q+k)^2] =iq_0^{\frac{1}{2}}\sum_{i,j}\int d^4x\, e^{ik\cdot x}(q-k)_\mu\epsilon_{3ij} \times\langle 0|[j_\mu(0),J_j(x)]\theta(-x_0)|q_i\rangle. \quad (3.29)$$

Finally, the factor $(q-k)_\mu$ can be replaced by a spatial derivative of the matrix element, since by gauge invariance $q_\mu M_\mu=-k_\mu M_\mu$. We shall not discuss the details but merely content ourselves with the remark that the derivation of the dispersion relations can be carried out by the method of Oehme.[16] The masses are such that where one is required to integrate the absorptive part of the amplitude over an unphysical region $(|k_0|<\mu)$, that part conveniently vanishes. One now readily establishes that the quantity $[-4\mu^2-(k+q)^2]M$, regarded as a function of k_0 in the rest frame of q, is analytic in the k_0 plane cut from μ to infinity. Assuming that M has no pole at $-(q+k)^2=4\mu^2$, we can write the once-subtracted dispersion relation

$$M[(q+k)^2]=1-\frac{(q+k)^2}{\pi} \times\int_{4\mu^2}^{\infty}d\xi'\frac{\text{Im}M(-\xi')}{\xi'[\xi'+(q+k)^2-i\epsilon]}. \quad (3.30)$$

To determine ImM, we write down the absorptive part, A_μ, of (3.28):

$$A_\mu=\pi q_0^{\frac{1}{2}}\sum_s\langle 0|j_\mu|s\rangle\langle s|J_j|q_i\rangle\delta(p_s-k-q). \quad (3.31)$$

As usual we have introduced a sum over a complete set of states; and in order to preserve the proper reality conditions at each stage of approximation we understand this to be one-half the sum over "in" and "out" states. The least massive state which can contribute is the two-pion state. Generally, states consisting only of pions must contain an even number of them. States with a pair of K mesons may contribute, as can the nucleon-pair state, etc. Suppose we limit our attention to the two-pion and the nucleon-pair intermediate states. Then in (3.31) we encounter $\langle 0|j_\mu|\pi\pi\rangle$, which leads us back to our amplitude M_μ, and $\langle 0|j_\mu|N\bar{N}\rangle$, which is the nucleon electromagnetic vertex function, aside from trivial factors. We thus generate a set of coupled integral equations which relate the meson and nucleon electromagnetic form factors.

Fortunately, it can be shown that the nucleon-pair state probably makes a very small contribution to $M[(q+k)^2]$, at least for $-(q+k)^2\lesssim 4m^2$. This follows from a unitarity argument of the sort we have employed

[16] R. Oehme, Nuovo cimento **10**, 1316 (1956).

previously. Thus, the nucleon-pair state contribution to A_μ is given by

$$A_\mu(\text{pair}) = \pi q_0^{\frac{1}{2}} \sum_{\text{spins}} \int \frac{d^3N d^3\bar{N}}{(2\pi)^3} \langle 0|j_\mu|N\bar{N}\rangle \times \langle N\bar{N}|J_j|q_i\rangle \delta(N+\bar{N}-q-k). \quad (3.32)$$

It is obvious that only the vector part of j_μ contributes here. The matrix element $\langle N\bar{N}|J_j|q_i\rangle$ describes the production of a nucleon pair in the collision of two pions; and one sees from the δ-function that we require this matrix element only in the physical region. Further, one sees by going into the rest system of the pair that only ${}^3S_1$ and ${}^3D_1$ pair states are involved here. Let us carry out the operations implied in (3.32) in this system.

The structure of $\langle 0|j_\mu|N\bar{N}\rangle$ in this system has already been given, in (3.16). For the matrix element $\langle N\bar{N}|J_j|q_i\rangle$ we write, in Pauli spinor notation,

$$\langle N\bar{N}\,\text{out}|J_j|q_i\rangle = -\left(\frac{m^2}{q_0N_0\bar{N}_0}\right)^{\frac{1}{2}} \chi_N{}^+\tfrac{1}{2}[\tau_i,\tau_j]\{B_S\boldsymbol{\sigma}\cdot(\mathbf{q}-\mathbf{k}) - B_D[3\boldsymbol{\sigma}\cdot\hat{N}\hat{N}\cdot(\mathbf{q}-\mathbf{k}) - \boldsymbol{\sigma}\cdot(\mathbf{q}-\mathbf{k})]\}\chi_{\bar{N}}, \quad (3.33)$$

where B_S and B_D are related to the ${}^3S_1$ and ${}^3D_1$ amplitudes, respectively, and are functions of N_0. Carrying out the integrations in (3.32), we find

$$(\text{Im}M)_{\text{pair}} = +\frac{\sqrt{2}}{\pi}\left(\frac{|\mathbf{N}|N_0}{2}\right) \times \text{Re}\Big\{\tfrac{2}{3}(B_S+B_D)(F_1{}^*+2mF_2{}^*)+\tfrac{1}{3}(B_S-2B_D) \times \left(\frac{m}{N_0}F_1{}^*+\frac{N_0}{m}2mF_2{}^*\right)\Big\}\theta(-\xi-4m^2); \quad (3.34)$$

where, in terms of the invariant variable $\xi=(N+\bar{N}^2)=(k+q)^2$ we have

$$|\mathbf{N}| = (-\tfrac{1}{4}\xi - m^2)^{\frac{1}{2}}, \qquad N_0 = (-\tfrac{1}{4}\xi)^{\frac{1}{2}}. \quad (3.35)$$

The unitarity limits on the amplitudes B_S and B_D are obtained by computing the total pair production cross section from (3.33) and demanding that this cross section be no larger than what is allowed by unitarity for pion-pion collisions in the $J=1$, $I=1$ state. We find

$$\frac{2|\mathbf{k}||\mathbf{N}|}{\pi}[|B_S|^2+2|B_D|^2] \leqslant \frac{3\pi}{|\mathbf{k}|^2}, \quad (3.36)$$

where $|N|$ is given by (3.35). In order to obtain a rough upper limit on ImM, we set the nucleon form factors in (3.34) equal to their static values: $F_1=e/2$; $F_2=(\mu_p-\mu_n)/2$. We further allow $|B_S|$ and $|B_D|$ to take on separately the maximum values permitted by (3.36), choosing algebraic signs to maximize ImM. Substituting ImM into the dispersion relation (3.30) and looking at the leading terms in an expansion in $(q+k)^2$, we find

$$M((q+k)^2) = 1 - \frac{1}{3m^2}(q+k)^2 + \cdots. \quad (3.37)$$

This result corresponds to a mean square radius of $2/m^2$ for the meson form factor; but our whole procedure has been such as to probably grossly overestimate this quantity. In any case, this estimated upper limit coming from the pair state is, as we shall see, probably small compared to the contribution from the two-pion intermediate state. Before turning to the latter, it is perhaps worth noting that if we had used perturbation theory to compute the matrix element $\langle N\bar{N}|J_j|q_i\rangle$ we would have found for the mean square radius the result $\langle r^2\rangle = 0.7/\mu^2$, which is about 16 times as large as the above result. Again, this is attributable to the fact that perturbation theory badly violates unitarity. Since the nucleon-pair state seems to make such a small contribution to the meson form factor, we shall ignore it from now on and retain only the two-pion state.

We have already noted that the two-pion state generates an integral equation for the form factor $M((q+k)^2)$. It is possible to discuss the structure of this in quite general terms and we shall begin in this way. Setting $\xi=(q+k)^2$, let us introduce a phase angle $\varphi(-\xi)$ according to

$$\text{Im}M(\xi) = \tan\varphi(-\xi)\ \text{Re}M(\xi)\theta(-\xi-4\mu^2), \quad (3.38)$$

where the step function appears because the lowest mass configuration which contributes to (3.31) is the two-pion state. Of course, (3.38) determines φ only to within an additive multiple of π. It is physically reasonable, however, to suppose that $\tan\varphi(-4\mu^2)=0$, and we shall make the convention that $\varphi(-4\mu^2)=0$. In place of (3.30) we can now write

$$M(\xi) = 1 - \frac{\xi}{\pi}\int_{4\mu^2}^{\infty} d\xi' \frac{\tan\varphi(\xi')\ \text{Re}M(-\xi')}{\xi'(\xi'+\xi-i\epsilon)}. \quad (3.39)$$

We now regard the function $\varphi(\xi)$ as a known quantity. Then Eq. (3.39) may be interpreted as an integral equation for $M(\xi)$, the general solution to which is easily given. The information about M which may be read from (3.39) is the following: (1) The function $M(\xi)$ can be extended to a function analytic in the ξ plane cut from $-\infty$ to $-4\mu^2$. (2) Just below the cut the real and imaginary parts are related according to (3.38). (3) M has the value unity at $\xi=0$. Evidently we seek a solution such that $\tan\varphi(\xi)\ \text{Re}M(-\xi)/\xi$ approaches zero since otherwise the equation as it stands is meaningless.

The mapping problem posed by Eq. (3.39) is a standard one and the general solution is[17]

$$M(\xi)=P(\xi)\exp\left\{-\frac{\xi}{\pi}\int_{4\mu^2}^{\infty}d\xi'\frac{\varphi(\xi')}{\xi'(\xi'+\xi-i\epsilon)}\right\}, \quad (3.40)$$

where $P(\xi)$ is as yet an arbitrary polynomial. As a matter of fact the more general problem where the 1 in (3.39) is replaced by a function of ξ may also be easily solved but we shall not go into the question here. The only restrictions which we may impose upon the polynomial $P(\xi)$ are that $P(0)=1$ and the degree must be no higher than would permit the existence of the integral in (3.39). This leaves, of course, a great deal of arbitrariness, in general, depending on the asymptotic form of $\varphi(\xi)$. It may also happen that $\varphi(\xi)$ is such that there is no solution to Eq. (3.39) even with $P(\xi)=1$; this would mean that the equation as it stands is meaningless and further subtractions (which would introduce new constants into the theory) would be required. We shall not consider this possibility further but assume in fact that solutions do exist and see to what extent they may be uniquely specified.

The existence of a multiplicity of solutions to dispersion equations like (3.39) is an old story, and the problem cannot be disposed of without supplying more information from the outside. One simple way to get rid of $P(\xi)$ is to assert that M has no zeros in the finite plane. There is no physical basis for such an assumption. What is equivalent to this, however, is the requirement that the solution (3.40) be chosen such as to agree with the *iteration* solution of (3.39). Thus we imagine that φ has a parameter of smallness associated with it and insist that the power series expansion of (3.40) agree with the series generated by the iteration solution of (3.39). This evidently leads to $P(\xi)=\text{const}=1$ and we take finally as our solution

$$M(\xi)=\exp\left\{-\frac{\xi}{\pi}\int_{4\mu^2}^{\infty}d\xi'\frac{\varphi(\xi')}{\xi'(\xi'+\xi-i\epsilon)}\right\}. \quad (3.41)$$

We shall discuss below, through an analysis of the two-pion contribution to (3.31), the physical meaning of $\varphi(\xi)$—at least for small ξ. For large ξ we have no simple physical interpretation of φ and have no feeling about its asymptotic behavior, which is, of course, crucial for the asymptotic value of M. It may be possible to prove by the method of Lehmann, Symanzik, and Zimmermann[18] that $M(\xi)$ in fact approaches zero at infinity and we shall assume that this is true. [A sufficient condition for this to happen is $\varphi(\xi)\to$ constant $=\varphi_0>0$; in this case $M(\xi)\to|\xi|^{-\varphi_0/\pi}$]. If $M(\xi)$ does approach zero at infinity, then since we require it in our dispersion integrals for only moderate ξ values, the precise asymptotic form of φ will not play an important role. In the model to be discussed below, φ is probably given quite reasonably in the interval $4\mu^2<\xi<30\mu^2$ and what happens beyond that is for our purposes not important.

In a related problem which we have discussed elsewhere by use of reaction matrix techniques,[11] it has been shown that in a situation of the sort under discussion we can express φ in the form

$$\tan\varphi=\frac{\mathrm{Re}(e^{i\delta}\sin\delta)}{1-\mathrm{Im}(e^{i\delta}\sin\delta)}, \quad (3.42)$$

where in the present problem δ is to be identified with the (complex) phase shift for pion-pion scattering in the state of angular momentum unity, isotopic spin unity $(J=1, I=1)$. To see how this comes about in dispersion theory, and to obtain some idea of the range of ξ for which it has validity, let us turn to the calculation of the absorptive part A_μ of (3.31). For values of ξ sufficiently small so that the main contributions come from the two-pion state, we have

$$A_\mu=\pi q_0^{\frac{1}{2}}\sum_{r,s}\int\frac{d^3p\,d^3l}{(2\pi)^3}\langle 0|j_\mu|p_rl_s\rangle \times\langle p_rl_s|J_j|q_i\rangle\delta(p+l-q-k), \quad (3.43)$$

where r and s are isotopic labels and p and l are the four-momenta of the intermediate pions. The first factor is just the vertex we are studying and is expressed in terms of the form factor M by (3.35), in the case of two-pion "in" states; for "out" states, one replaces M by M^*. Recall that half the sum of "in" and "out" states is implied in (3.43). The second factor in (3.43) is proportional to the scattering amplitude for pion-pion scattering and it is clear that we are always in the physical region. Obviously $A_0=0$ in the rest frame of the pions; and since j transforms like a vector it is evident that only two-pion $J=1$ states contribute. It is also evident that only the isotopic-spin-one states are relevant here. The matrix element is then completely characterized by the complex phase shift δ for pion-pion scattering in the $J=1$, $I=1$ state. The precise relation is as follows:

$$\langle p_rl_s\,\text{out}|J_j|q_i\rangle=(8p_0l_0q_0)^{-\frac{1}{2}}\frac{4\pi}{\sqrt{2}}\frac{(-\xi)^{\frac{1}{2}}}{(-\frac{1}{4}\xi-\mu^2)^{\frac{3}{2}}} \times 3e^{i\delta}\sin\delta\left(\frac{\delta_{ri}\delta_{sj}-\delta_{rj}\delta_{si}}{2}\right)(p-l)\cdot(q-k), \quad (3.44)$$

where $\xi=(q+k)^2$ and δ is to be regarded as a function of ξ through its dependence on the center-of-mass wave number $(-\frac{1}{4}\xi-\mu^2)^{\frac{1}{2}}$. For an "in" state the right-hand side of (3.44) would be replaced by its complex conjugate.

We now have all the elements required for the evaluation of (3.43), and hence of the two-pion contribution

[17] N. I. Muskhelishvili, *Singular Integral Equations* (P. Boordhoff N. V., Groningen, Holland, 1953).

[18] Lehmann, Symanzik, and Zimmermann, Nuovo cimento 2, 425 (1955).

to $\mathrm{Im}M$. We find the result

$$\mathrm{Im}M(\xi)=\mathrm{Re}(M^*e^{i\delta}\sin\delta)\theta(-\xi-4\mu^2), \tag{3.45}$$

hence

$$\mathrm{Im}M(\xi)=\frac{\mathrm{Re}(e^{i\delta}\sin\delta)}{1-\mathrm{Im}(e^{i\delta}\sin\delta)}\mathrm{Re}M\theta(-\xi-4\mu^2). \tag{3.46}$$

This is precisely the result stated in (3.42). The argument given in reference 11 makes it plausible, as we also see here, to retain a complex value of δ even though we have dropped states other than the two-pion state. The point is that other states could be relatively unimportant in contributing to $\mathrm{Im}M$, at least for small ξ, even if they play an important role in pion-pion scattering when inelastic processes compete with the scattering. Of course, δ is real below the threshold for inelastic processes.

Needless to say, we have no experimental—or theoretical—information on the pion-pion scattering phase shift. However, since the scattering presumably takes place through virtual baryon pairs it is reasonable to assume that the "range" of the interaction is small. Consequently we propose to represent the energy dependence of the phase shift by a scattering length approximation; namely, with k the center-of-mass wave number, we take

$$\tan\delta(-\xi)=k^3a^3=(-\tfrac{1}{4}\xi-\mu^2)^{\frac{3}{2}}a^3, \tag{3.47}$$

where a is the scattering length, expected to be of order m^{-1}. Of course, the representation of (3.47) makes no sense for large k, when inelastic processes can seriously compete with scattering and cause δ to become complex. But as already said, we are only concerned with the behavior of the form factor M for not too large ξ insofar as we restrict our attention to the low momentum-transfer structure of the nucleon.

Because in our model the phase shift δ is taken real, we see from (3.42) that the phase φ is just identical

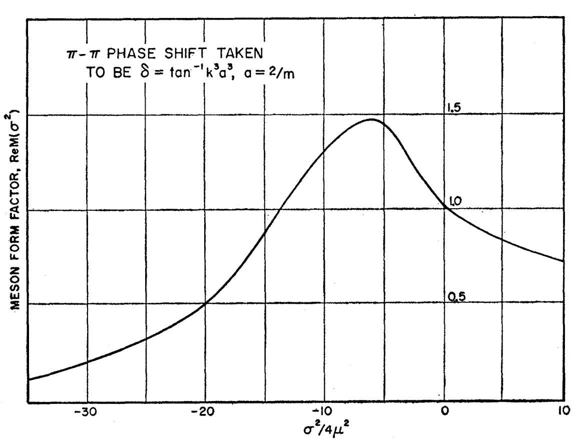

FIG. 1. Real part of the pion form factor; the pion-pion scattering phase shift is taken to be $\delta=\tan^{-1}(k^3a^3)$, where k is the wave number in the center-of-mass system and a is the scattering length chosen to be $2/m$.

with δ. We can now evaluate (3.41). Setting $\alpha=\mu a$ and $y=\xi/4\mu^2$, we find

$$\mathrm{Re}M(y)=M(y)=\frac{1+\alpha+\alpha^2}{1+\alpha}\left(\frac{\alpha^2(y+1)-1}{[\alpha^2(y+1)]^{\frac{3}{2}}-1}\right),\quad -1<y<\infty, \tag{3.48a}$$

$$\mathrm{Re}M(y)=\frac{1+\alpha+\alpha^2}{1+\alpha}\left(\frac{\alpha^2(-y-1)+1}{[\alpha^2(-y-1)]^3+1}\right),\quad y<-1, \tag{3.48b}$$

$$\mathrm{Im}M(y)=\frac{1+\alpha+\alpha^2}{1+\alpha}[\alpha^2(-y-1)]^{\frac{3}{2}}\times\left(\frac{\alpha^2(-y-1)+1}{[\alpha^2(-y-1)]^3+1}\right),\quad y<-1. \tag{3.48c}$$

We see that for large positive y, $M(y)\sim y^{-\frac{1}{2}}$; for large negative y, $\mathrm{Re}M(y)\sim(-y)^{-2}$, $\mathrm{Im}M(y)\sim(-y)^{-\frac{1}{2}}$.

For the mean square charge radius of the meson we find the result

$$\langle r^2\rangle=\frac{3}{4\mu^2}\alpha^2\left(\frac{\alpha^3+\alpha^2-\alpha+2}{(\alpha+1)(\alpha^4+\alpha^2+1)}\right). \tag{3.49}$$

There is no need to emphasize that this quantity is not at present accessible to direct measurement. For illustration, we plot in Fig. 1 the function $\mathrm{Re}M(y)$ for the case $a=2/m$, hence $\alpha\approx0.3$. For this choice, $\langle r^2\rangle=0.08/\mu^2$, which is twice as large as the upper bound on the contribution from the nucleon-pair state.

F. Quantitative Summary

Although our evaluation of the meson form factor M has no validity for large ξ, it is clear that if M really does vanish asymptotically then the convergence properties of the dispersion integrals for the isotopic vector nucleon form factors would be greatly improved. This raises the more general question as to whether it is really necessary to use a subtracted dispersion relation for the isotopic vector form factor $F_1{}^V$, as we have done. If it is assumed that no subtraction need be made, one in effect assumes that he can compute the charge $2F_1{}^V(0)$. This point has been discussed by Chew,[19] who carries out the computation by determining $\mathrm{Im}F_1{}^V(-\sigma^2)$ in perturbation theory and cutting off the dispersion integral at $4m^2$. In this way he finds $2F_1{}^V(0)=1.28e$. Taking into account rescattering we find $2F_1{}^V(0)\approx0$. It may be that the rescattering estimates are unreliable, or that higher mass configurations play an important role. The matter evidently cannot easily be decided. We prefer in the absence of trustworthy calculations to adopt a subtracted dispersion relation.

We now return to the main question, which is to introduce the structure of the meson into the problem

[19] G. F. Chew (to be published).

of nucleon structure. We carry out the evaluation of the nucleon isotopic vector form factors using (3.48) together with (3.10) and (3.11). The rescattering corrections are treated as before. In the dispersion integrals we drop contributions from $(\text{mass})^2$ values larger than $4m^2$, since we have already seen from unitarity arguments alone that such contributions are small. For illustration we again take the pion-pion scattering length to be $a=2/m$. In Table I we exhibit the whole numerical situation for the contributions from the two-pion state. Qualitatively, the situation can be summarized in the following way: the isotopic vector magnetic moment and magnetization density radius seem to be reasonably well accounted for in terms of the two-pion state contributions. The charge density radius on the other hand turns out to be far too small relative to the experimental value. It may be that this quantity is peculiarly sensitive to our rescattering correction approximation; or what is equally likely (assuming the experiments are correct!) higher configurations are playing an unexpectedly important role. It is amusing to note that if the charge density radius is computed under the assumption that we have no subtraction in the dispersion relation for $F_1{}^V$ [and hence is given by $-6F_1{}^{V\prime}(0)/F_1{}^V(0)$ instead of $-12F_1{}^{V\prime}(0)/e$] we would get a very large value, since $F_1{}^V(0)\simeq 0$ according to our estimates.

G. Three-Pion State

The least massive state which contributes to the isotopic scalar properties of nucleons is the three-pion state; and one would expect this state to be the most important one in determining the isotopic scalar magnetic moment and mean square radii. Unfortunately, we are unable to make any even crude quantitative estimates of the effects from the three-pion state. Even a perturbation calculation (lowest order $\sim eg^6$ is prohibitive.

What we shall do, therefore, is merely carry out an analysis which separates off purely kinematic effects and exposes the basic structure of the problem. From (2.9) we see that what is involved here is the product $\langle 0|j_\mu|3\pi\rangle\langle 3\pi|f|p\rangle$. In the rest frame of the nucleon pair

TABLE I. Summary of the two-pion contributions to the isotopic vector magnetic moment μ_V, magnetization density mean square radius $\langle(r_2{}^2)_V\rangle$, and charge density mean square radius $\langle(r_1{}^2)_V\rangle$. In the first column the quantities in parentheses denote the limits of integration in the appropriate dispersion integrals.

	μ_V	$\langle(r_2{}^2)_V\rangle$	$\langle(r_1{}^2)_V\rangle$
Pert. theory $(4\mu^2\to\infty)$	$1.67\ e/2m$	$0.12/\mu^2$	$0.24/\mu^2$
Pert. theory $(4\mu^2\to 4m^2)$	$0.87\ e/2m$	$0.22/\mu^2$	$0.19/\mu^2$
Pert. theory+pion form factor $(4\mu^2\to 4m^2)$	$0.77\ e/2m$	$0.28/\mu^2$	$0.20/\mu^2$
Pert. theory+rescattering $(4\mu^2\to 4m^2)$	$1.90\ e/2m$	$0.16/\mu^2$	$0.03/\mu^2$
Pert. theory+rescattering +pion form factor $(4\mu^2\to 4m^2)$	$1.39\ e/2m$	$0.23/\mu^2$	$0.07/\mu^2$

(which is, of course, also the rest frame of the three-pion system), the first matrix element describes the process of a virtual photon producing three pions in a state of total angular momentum unity. The second matrix element describes the process of pair annihilation into three mesons. One sees that only the ${}^3S_1$ and ${}^3D_1$ pair states are involved here. From charge-conjugation invariance it follows that the three pions must have different charges (i.e., $|3\pi\rangle=|\pi^+\pi^-\pi^0\rangle$). This, together with gauge invariance, $(p^++p^-+p^0)_\mu\langle 0|j_\mu|p^+p^-p^0\rangle=0$, leads to the structure of the matrix element, which must transform like a pseudovector, given by

$$\langle 0|j_\mu|p^+p^-p^0\text{ out}\rangle = -i(8w_+w_-w_0)^{-\frac{1}{2}}\epsilon_{\mu\nu\lambda\sigma}p_\nu{}^+p_\lambda{}^-p_\sigma{}^0H^*, \quad (3.50)$$

where H is a scalar function of the momenta and the pion energies have been denoted by w.

In the rest system of the three pions let us denote by $\mathbf{k}$ the relative momentum of π^+ and π^- and by $\mathbf{q}$ the momentum of π^0. Let $2E$ denote the total center-of-mass energy, so that E is the energy of either member of the nucleon pair. Then in the rest frame $-i\epsilon_{\mu\nu\lambda\sigma}p_\nu{}^+p_\lambda{}^-p_\sigma{}^0$ reduces to $2E(\mathbf{k}\times\mathbf{q})_\mu$, $\mu=1, 2, 3$ ($\mu=4$ does not contribute). In terms of the well-known Dalitz description,[20] in which the relative angular momentum of (π^+,π^-) is denoted by l and that of π^0 relative to the center of mass of (π^+,π^-) denoted by L, we have $l=L=$ odd integer. In the rest frame we then have

$$\langle 0|\mathbf{j}|p^+p^-p^0\text{ out}\rangle = 2E(\mathbf{k}\times\mathbf{q})(8w_+w_-w_0)^{-\frac{1}{2}}H^*(k^2,q^2,\lambda^2), \quad (3.51)$$

where

$$\lambda^2=\mathbf{k}\cdot\mathbf{q}/|\mathbf{k}|\,|\mathbf{q}|.$$

The other matrix element which we require is that describing pair annihilation into three pions. Retaining only the contributions from pairs in the ${}^3S_1$ and ${}^3D_1$ states, we have in the center-of-mass system

$$\bar{v}(\bar{p})\langle p^+p^-p^0\text{ out}|f|p\rangle = -(m/E)^{\frac{1}{2}}(8w_+w_-w_0)^{-\frac{1}{2}}\pi_{\bar{p}}{}^*\{\alpha\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{q} - (\beta/\sqrt{2})[3\boldsymbol{\sigma}\cdot\hat{p}\hat{p}\cdot(\mathbf{k}\times\mathbf{q})-\sigma(\mathbf{k}\times\mathbf{q})]\}\chi_p, \quad (3.52)$$

where α and β are, respectively, amplitudes for three-pion production by nucleon pairs in the ${}^3S_1$ and ${}^3D_1$ states. They of course depend on the variables k^2, q^2, λ^2.

The expressions (3.51) and (3.52) are now to be inserted into (2.9). Some of the integrations can be carried out explicitly and one finds for the contribution from the three-pion state, in the rest system,

$$\mathbf{A}^J(3\pi)=+\frac{4E}{3(2\pi)^3}\int_0^{q(E)}q^2dq\int_{\kappa(q,E)}^{K(q,E)}k^2dk\left(\frac{1-\lambda^2}{\lambda^2}\right)\times\left(\frac{2E-(\mu^2+k^2)^{\frac{1}{2}}}{(\mu^2+k^2)^{\frac{1}{2}}}\right)\chi_{\bar{p}}{}^*\bigg\{\boldsymbol{\sigma}\,\mathrm{Re}(H^*\alpha) + (3\boldsymbol{\sigma}\cdot\hat{p}\hat{p}-\boldsymbol{\sigma})\frac{1}{\sqrt{2}}\,\mathrm{Re}(H^*\beta)\bigg\}\chi_p; \quad (3.53)$$

[20] R. H. Dalitz, Phys. Rev. 94, 1046 (1954).

where

$$q(E)=[(E-\tfrac{1}{2}\mu)^2-\mu^2]^{\frac{1}{2}},$$

$$K(q,E)=\left[\frac{(3\mu^2+4q^2-4E^2)^2}{16E^2}-\mu^2\right]^{\frac{1}{2}},$$

$$\lambda^2=\frac{1}{k^2q^2}\left\{\left(\mu^2+q^2+\frac{k^2}{4}\right)^2 -\frac{1}{4}\left[[2E-(\mu^2+k^2)^{\frac{1}{2}}]^2-2\left(\mu^2+q^2+\frac{k^2}{4}\right)\right]^2\right\}, \quad (3.54)$$

and $\kappa(q,E)$ is the value of k which corresponds to $\lambda=1$ if such exists; otherwise $\kappa=0$. Finally, for the imaginary parts of the isotopic scalar form factors we obtain the results

$$\mathrm{Im}F_1{}^S[(p+\bar{p})^2]=-\frac{4m}{3(2\pi)^3}\int_0^{q(E)}q^2dq \times\int_{\kappa(q,E)}^{K(q,E)}k^2dk\left(\frac{1-\lambda^2}{\lambda^2}\right)\left(\frac{2E-(\mu^2+k^2)^{\frac{1}{2}}}{(\mu^2+k^2)^{\frac{1}{2}}}\right) \times\mathrm{Re}\left\{\frac{E}{E+m}H^*\left(\alpha+\frac{1}{\sqrt{2}}\beta\right)+\frac{3mE}{E^2-m^2}\frac{1}{\sqrt{2}}H^*\beta\right\}; \quad (3.55)$$

$$\mathrm{Im}F_2{}^S[(p+\bar{p})^2]=-\frac{4m}{3(2\pi)^3}\int_0^{q(E)}q^2dq\int_{\kappa(q,E)}^{K(q,E)}k^2dk \times\left(\frac{1-\lambda^2}{\lambda^2}\right)\left(\frac{2E-(\mu^2+k^2)^{\frac{1}{2}}}{(\mu^2+k^2)^{\frac{1}{2}}}\right)\mathrm{Re}\left\{\frac{H^*}{2(E+m)}\left(\alpha+\frac{1}{\sqrt{2}}\beta\right) -\frac{3E}{2(E^2-m^2)}\frac{1}{\sqrt{2}}H^*\beta\right\}; \quad (3.56)$$

and $E^2=-\frac{1}{4}(p+\bar{p})^2$. We have not written it in, but each of the above expressions should be multiplied by the step function $\theta[-(p+\bar{p})^2-(3\mu)^2]$.

It is scarcely necessary to remark that one cannot obtain any quantitative impressions from these results. There are sufficiently many unknown functions here so that one could easily arrange it to produce a large isotopic scalar charge radius and at the same time a very small magnetic moment. It is unfortunately the case that the three-pion state, which is perhaps the most significant contributor to the mysterious isotopic scalar properties of the nucleon, cannot be treated quantitatively without prohibitive labor. We remark again that even a perturbation calculation would be very worthwhile.

H. K Meson-Pair State Contributions

It was suggested some time ago by Sandri[21] that the electromagnetic structure of the nucleon would be influenced by the interaction between nucleons and strange particles. In particular he considered the coupling of K mesons to nucleons on the basis of a cutoff model and found rather sizable effects. More recently, relativistic perturbation theory calculations have been carried out along these lines.[22] For reasons to be discussed in Sec. IV in connection with nucleon pair intermediate states we feel that such calculations are meaningless. What might be more reasonable [depending on the size of K-nucleon-hyperon coupling constants] is to treat the contribution of the K-meson current to nucleon structure by perturbation theory and neglect that coming from the intermediate hyperon current. In view of the fact that $g_K{}^2<g_\pi{}^2$, the use of perturbation theory may be more reliable in the present instance than it was for the pion current.

The algebra of the $|2K\rangle$ contribution is essentially the same as that for the $|2\pi\rangle$ state; the only difference is that the mass of the intermediate hyperon, a Λ or a Σ, is different from that of the nucleon. There is, of course, also the question of the parity of the K meson relative to baryons. We have done the calculation for both cases. For the proton, there are two diagrams, corresponding to $p\to(\Sigma_0,\Lambda_0)+K_+\to p$; whereas for the neutron we have only $n\to\Sigma^-+K_+\to n$ in lowest order. The couplings are taken in the form given by Gell-Mann.[23]

Using for convenience the same mass for Σ and Λ, chosen intermediate between the two actual masses, we obtain the numerical results shown in Table II. If, as has been suggested by Schwinger,[24] $g_\Lambda{}^2=g_\Sigma{}^2$, we have only isotopic scalar contributions. It is impossible, however, for any value of $g_s{}^2$ or choice of parity to obtain both a large charge mean square radius and a small anomalous moment. For example, if $g_s{}^2\sim1$ and the K meson is pseudoscalar, the moment is of reasonable size (but wrong sign) but the charge radius is negligibly small. If the K meson is scalar and $g_s{}^2\sim15$, the charge radius is reasonable ($\sim0.2/\mu^2$) but the moment is absurd ($\sim2.4e/2m$).

We conclude that K mesons very likely do not play any important role in nucleon electromagnetic structure, unless the coupling constants turn out to be so large that, here as in the pion case, rescattering corrections to perturbation theory are very important.

TABLE II. Contribution of K-meson current to nucleon structure. To obtain the isotopic scalar and vector contributions, in the above table set $g^2=g_S{}^2=(3g_\Sigma{}^2+g_\Lambda{}^2)/16$ or $g^2=g_V{}^2=(g_\Lambda{}^2-g_\Sigma{}^2)/16$, respectively.

	Pseudoscalar coupling	Scalar coupling
$F_2(0)$	$g^2(0.0573)e/2m$	$-g^2(0.16)e/2m$
$F_2'(0)$	$-g^2(0.0074)e/2m^3$	$g^2(0.030)e/2m^3$
$F_1'(0)$	$-g^2(0.0101)e/m^3$	$-g^2(0.0475)e/m^3$

[21] G. Sandri, Phys. Rev. **101**, 1616 (1956).

[22] Y. Nogami, Nuovo cimento **4**, 985 (1957).

[23] M. Gell-Mann, Phys. Rev. **106**, 1296 (1957).

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IV. ROLE OF NUCLEON-ANTINUCLEON PAIRS

The last topic we shall treat is the role of intermediate nucleon-antinucleon pair states in the electromagnetic structure problem. In lowest order perturbation theory such states appear simultaneously and on essentially an equal footing with the two-pion state. It has long been recognized that in perturbation theory the pairs make an unreasonably large contribution to the anomalous moment; there have been frequent speculations that a more accurate treatment would correct this situation.

The results of lowest order perturbation theory, calculated either directly by Feynman methods or from the dispersion formula (4.2) [into which one inserts the lowest order amplitudes], are as follows:

$$\tfrac{2}{3}F_2{}^S(0)=-2F_2{}^V(0)=-\frac{e}{2m}\left(\frac{g^2}{4\pi}\right)\frac{1}{4\pi}\times\left[(1+2\eta)+\eta(1-\eta)\ln\eta+\frac{\eta^{\frac{3}{2}}(1-3\eta)}{(1-\eta/4)^{\frac{1}{2}}}\cos^{-1}\left(\frac{\eta^{\frac{1}{2}}}{2}\right)\right]=-1.13e/2m;$$

$$\tfrac{2}{3}F_2{}'^S(0)=-2F_2{}'^V(0)=-\frac{e}{2m}\cdot\left(\frac{1}{m^2}\right)\left(\frac{g^2}{4\pi}\right)\frac{1}{8\pi}\times\left[\frac{2\eta^2-(19/3)\eta-\frac{4}{3}}{4-\eta}+(\eta^2-\tfrac{2}{3}\eta)\ln\eta+\left(20-\frac{40}{3}\eta+2\eta^2\right)\left(\frac{\eta}{4-\eta}\right)^{\frac{3}{2}}\cos^{-1}\left(\frac{\eta^{\frac{1}{2}}}{2}\right)\right]=0.181e/2m^3;$$

$$\tfrac{2}{3}G_1{}'^S(0)=-2G_1{}'^V(0)=-\frac{e}{m^2}\left(\frac{g^2}{4\pi}\right)\frac{1}{16\pi}\times\left[\frac{4-5\eta+2\eta^2}{3(4-\eta)}+\frac{\eta^2}{3}\ln\eta+\left(\frac{\eta}{4-\eta}\right)^{\frac{3}{2}}(4-4\eta+\tfrac{2}{3}\eta^2)\cos^{-1}\left(\frac{\eta^{\frac{1}{2}}}{2}\right)\right]=-0.1e/m^3. \quad (4.1)$$

The anomalous moment is seen to be quite large, whereas the contributions to the mean square radii are seen to be negligible.

The dispersion-theoretic treatment of the pair state proceeds from Eq. (2.9) where the state $|s\rangle$ is taken as $|n\bar{n}\rangle$. We have

$$A_\mu{}^J(N\bar{N})=-\pi\left(\frac{p_0}{m}\right)^{\frac{1}{2}}\sum_{\text{spins}}\int\frac{d^3n\,d^3\bar{n}}{(2\pi)^3}\bar{v}(\bar{p})\times\langle 0|j_\mu|n\bar{n}\rangle\langle n\bar{n}|f|p\rangle\delta(n+\bar{n}-p-\bar{p}), \quad (4.2)$$

where, as usual, we take half the sum of "in" and "out" pair states. As we have described in Sec. II, $\bar{v}(\bar{p})\langle n\bar{n}|f|p\rangle$ is proportional to the amplitude for nucleon-antinucleon scattering. The first factor, $\langle 0|j_\mu|n\bar{n}\rangle$, is the very quantity we are studying; just as in the case of the pion vertex (see Sec. III), we are generating an integral equation. In Appendix A we discuss the solution of this equation for the isotopic scalar quantities, $F_1{}^S$ and $F_2{}^S$, when the nucleon-antinucleon amplitude is treated in lowest order perturbation theory. This is a sort of ladder approximation similar to a treatment given by Edwards. We find that although this represents a great improvement over perturbation theory it is probably not a very accurate approximation.

The evaluation of (4.2) is most conveniently carried out in the rest frame of $|n\bar{n}\rangle$. As we have discussed several times before, one need then consider only ${}^3S_1$ and ${}^3D_1$ intermediate pair states, and only the nucleon-antinucleon amplitudes leading to these final states are required. A certain amount of caution must be exercised in expressing the scattering amplitudes in terms of two-component spinors since we have been using negative-energy four-component spinors, $v(\bar{p})$, in our discussion instead of true antiparticle quantities. In particular, for a matrix element in Pauli spin space such as $\langle\bar{n}|Q|\bar{p}\rangle$ we must for consistency write $\chi_{\bar{p}}{}^*\sigma^{(2)}Q^T\sigma^{(2)}\chi_{\bar{n}}$ where the χ's are the usual two-component spinors. This has the effect of changing the signs of all quantities linear in $\boldsymbol{\sigma}$ since $\sigma^{(2)}\boldsymbol{\sigma}^T\sigma^{(2)}=-\boldsymbol{\sigma}$.

We write then for the relevant part of the nucleon-antinucleon scattering amplitude, in the rest system $\mathbf{p}+\bar{\mathbf{p}}=\mathbf{n}+\bar{\mathbf{n}}=0$,

$$\bar{v}(\bar{p})\langle n\bar{n},\text{out}|f|p\rangle=-\frac{4\pi p_0}{m^2|\mathbf{p}|}\left(\frac{m}{p_0}\right)^{\frac{3}{2}}\left[\beta_S\frac{3+\boldsymbol{\sigma}_1\cdot(-\boldsymbol{\sigma}_2)}{4}+\frac{\beta_{SD}}{8^{\frac{1}{2}}}\left\{\frac{3\boldsymbol{\sigma}_1\cdot\mathbf{n}(-\boldsymbol{\sigma}_2\cdot\mathbf{n})}{|\mathbf{n}|^2}-\boldsymbol{\sigma}_1\cdot(-\boldsymbol{\sigma}_2)+\frac{3\boldsymbol{\sigma}_1\cdot\mathbf{p}(-\boldsymbol{\sigma}_2\cdot\mathbf{p})}{|\mathbf{p}|^2}-\boldsymbol{\sigma}_1\cdot(-\boldsymbol{\sigma}_2)\right\}+\frac{\beta_D}{8}\left\{\left[\frac{3\boldsymbol{\sigma}_1\cdot\mathbf{n}(-\boldsymbol{\sigma}_2\cdot\mathbf{n})}{|\mathbf{n}|^2}-\boldsymbol{\sigma}_1\cdot(-\boldsymbol{\sigma}_2)\right]\times\left[\frac{3\boldsymbol{\sigma}_1\cdot\mathbf{p}(-\boldsymbol{\sigma}_2\cdot\mathbf{p})}{|\mathbf{p}|^2}-\boldsymbol{\sigma}_1\cdot(-\boldsymbol{\sigma}_2)\right]\right\}^T\right], \quad (4.3)$$

where

$$\begin{aligned}\boldsymbol{\sigma}_1&=\chi_n{}^*\boldsymbol{\sigma}\chi_p,\\ \boldsymbol{\sigma}_2&=\chi_{\bar{p}}{}^*\boldsymbol{\sigma}\chi_{\bar{n}}\\ 1&=\chi_n{}^*\chi_p\chi_{\bar{p}}{}^*\chi_{\bar{n}}.\end{aligned} \quad (4.4)$$

The over-all minus sign in (4.3) as well as the factor $4\pi p_0/m^2|\mathbf{p}|$ are inserted so that the β's have the familiar scattering-matrix significance. For example, $\beta_S=[\exp 2i\delta_s-1]/2i$ with δ_s the complex phase shift describing $S-S$ scattering. The unitarity restriction states that $|\beta_S|\leqslant 1$, $|\beta_{SD}|\leqslant 1/2$, $|\beta_D|\leqslant 1$. The minus sign that appears with $\boldsymbol{\sigma}_2$ in the various projection operators has been explained above; the superscript T on the last curly brace in (4.3) means that the $\boldsymbol{\sigma}_2$ operators are to be written in the order opposite to their appearance. The expression is then reduced to a linear function of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ and the result interpreted according to (4.4). For $\langle 0|j_\mu|n\bar{n}\rangle$ we use the form given in Eq. (3.16).

The evaluation of (4.2) with these expressions is elementary. Still in the rest system, $\mathbf{p}+\bar{\mathbf{p}}=0$, we have

$$\frac{m}{p_0}\mathbf{A}^J(N\bar{N})=-\chi_{\bar{p}}{}^*\boldsymbol{\sigma}\chi_p \operatorname{Re}\left[\beta_S\left(A^*+\frac{B^*}{3}\right)+\frac{\beta_{SD}}{\sqrt{2}}\left(A^*-\frac{B^*}{3}\right)-\frac{\beta_D}{3}B^*\right]-\chi_{\bar{p}}{}^*\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})\mathbf{p}}{|\mathbf{p}|^2}\chi_p\times\operatorname{Re}\left[-\frac{3\beta_{SD}}{\sqrt{2}}\left(A^*+\frac{B^*}{3}\right)+\beta_D B^*\right], \quad (4.5)$$

where

$$A=F_1+2mF_2,$$
$$B=\left[\frac{2mF_2}{mp_0}-\frac{F_1+2mF_2}{p_0(p_0+m)}\right]|\mathbf{p}|^2. \quad (4.6)$$

Comparison of (4.5) with (2.9) yields for the pair state contribution to $\operatorname{Im}F_1$ and $\operatorname{Im}F_2$ the results

$$\operatorname{Im}F_1=\frac{p_0}{p_0+m}C-\frac{mp_0}{|\mathbf{p}|^2}D,$$
$$\operatorname{Im}F_2=\frac{1}{2(p_0+m)}C+\frac{p_0}{2|\mathbf{p}|^2}D, \quad (4.7)$$

where

$$C=\operatorname{Re}\left[\beta_S\left(A^*+\frac{B^*}{3}\right)+\frac{\beta_{SD}}{\sqrt{2}}\left(A^*-\frac{B^*}{3}\right)-\frac{\beta_D}{3}B^*\right], \quad (4.8)$$
$$D=\operatorname{Re}\left[-3\frac{\beta_{SD}}{\sqrt{2}}\left(A^*+\frac{B^*}{3}\right)+\beta_D B^*\right].$$

For substitution into the dispersion relations we must imagine all quantities to be expressed in terms of the invariant variable $-(p+\bar{p})^2$ according to $p_0{}^2=-\frac{1}{4}(p+\bar{p})^2$, $|\mathbf{p}|^2=-\frac{1}{4}(p+\bar{p})^2-m^2$.

We have not kept track of the isotopic spin labels in the above equations, but these are easily inserted now. The scattering amplitudes are expressed in terms of isotopic scalar and vector parts (i.e., $I=0, I=1$) according to

$$\beta_S=\tfrac{1}{4}(3-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\beta_S{}^V+\tfrac{1}{4}(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\beta_S{}^S, \quad (4.9)$$

and similarly for β_{SD} and β_D. [The unusual signs in the projection operators have the same origin as those appearing with $\boldsymbol{\sigma}_2$ in (4.3).] The sum over intermediate isotopic spin states in (4.2) leads to

$$(j_\mu{}^S+\tau_3 j_\mu{}^V)(\tfrac{3}{4}\beta_S{}^V+\tfrac{1}{4}\beta_S{}^S)+\tau_i(j_\mu{}^S+\tau_3 j_\mu{}^V)\tau_i(\tfrac{1}{4}\beta_S{}^S-\tfrac{1}{4}\beta_S{}^V)=\beta_S{}^S j_\mu{}^S+\beta_S{}^V\tau_3 j_\mu{}^V. \quad (4.10)$$

Thus the isotopic vector and scalar parts of $\operatorname{Im}F_1$ and $\operatorname{Im}F_2$ are obtained by taking the corresponding isotopic amplitudes β in Eqs. (4.6–4.9).

We confine our attention to the isotopic scalar quantities since these are not coupled to the two-pion contributions discussed earlier. Substitution of Eq. (4.7) into the dispersion relations (2.5) leads one to a very complicated system of two coupled integral equations. This system may be reduced to a single Fredholm equation about whose solution little can be said without very much effort. Rather than attempt this, we have made only rough estimates of the contributions to the magnetic moment and charge radius.

The following rather drastic assumptions will be made: (a) Only the amplitude β_S will be retained; inclusion of β_{SD} and β_D should not modify the results appreciably. We assume that β_S has the maximum value allowed by unitarity, independent of energy. One would in fact expect it to be much smaller since there is an appreciable annihilation probability. (b) In computing $\operatorname{Im}F_1{}^S$ and $\operatorname{Im}F_2{}^S$ from (4.6) we shall replace F_1 and F_2 (which occur in A and B) on the right-hand side equal to their zero momentum transfer values, namely $F_1{}^S(0)=e/2$, $F_2{}^S(0)\approx 0$. Furthermore we neglect B in comparison with A. With these approximations, only the real part of β_S enters and this is set equal to one-half. Substituting into the dispersion integrals (2.5), we find for the pair state contributions

$$|F_1'^S(0)|\leqslant\frac{1}{\pi}\int_{4m^2}^{\infty}d\xi\,\xi^{-2}\frac{p_0}{p_0+m}\times\frac{1}{2}\times\frac{e}{2}=\frac{e}{8\pi m^2}(1-\ln 2)=\frac{0.012e}{m^2},$$
$$|F_2{}^S(0)|\leqslant\frac{1}{\pi}\int_{4m^2}^{\infty}d\xi\,\xi^{-1}\frac{1}{2(p_0+m)}\times\frac{1}{2}\times\frac{e}{2}=\frac{\ln 2}{2\pi}\times\frac{e}{2m}=0.11\frac{e}{2m}, \quad (4.11)$$

where $p_0{}^2=\xi/4$ in these integrals. We conclude from these estimates that the pair plays a numerically unimportant role in nucleon structure, at least for small momentum transfers.

There are a few points in connection with the above calculations which should be discussed further. In particular the replacement of the F's on the right-hand side of (4.6) by their static values requires comment. First let us assume that there are states less massive than the nucleon-pair state (e.g., three pions) which contribute appreciably to isotopic scalar quantities. By analogy with the behavior of the pion vertex $\langle 0|j_\mu|\pi\pi\rangle$ treated in Sec. III, we would expect the $F_1(-\xi)$ and $F_2(-\xi)$ in (4.6) to rise from their static values and then fairly soon after the threshold of the important masses (say $\xi\sim 9\mu^2$) decreases strongly; their value at $\xi=4m^2$, which is the lower limit for the nucleon pair contribution, should be much less than the static value, so that our rough procedure of using the static values should be a gross overestimate. If, on the other hand, the nucleon-pair state itself is the first significant source of isotopic scalar, the fact that the $F(-4m^2)$ might be significantly larger than the $F(0)$ might cause some concern over the validity of our estimate. We have studied this question in some detail and, using various models for the F's discussed in another connection,[11] have found that in spite of the fact that $F(-4m^2)>F(0)$ (as much as seven times larger in one example) and that, for $\xi>4m^2$, $F(-\xi)$ rises further before falling, there are no appreciable modifications of our estimates. The point is that the width of the peak in $F(-\xi)$, described as ξ increases, is very small and that much of the large ξ contributions to the values given in (4.11) are lost by the fact that $F(-\xi)$ approaches zero quite rapidly in all of the models treated. (The higher the maximum the more rapid is the decrease for large ξ.)

We have further studied certain special models of nucleon-antinucleon scattering for which the coupled integral equations described above could be exactly solved. For example, if one sets $\beta_D=0$ and $\beta_{SD}/\sqrt{2}=\beta_S$ and forms $G_1=F_1+2mF_2$, the integral equations for G_1 and F_2 decouple. The solutions are too complicated to exhibit although they are readily obtained. Rough evaluations of these based on reasonable assumptions for β_S, such as used in reference 11 lead to contributions which lie within the estimates given in (4.11). The perturbation example treated in Appendix A is characterized by the relations $(\beta_S-\beta_D)=\beta_{SD}/\sqrt{2}$ and $\beta_D=\beta_S/2$. The latter condition is very pathological, of course; in addition, the numerical value given to β_S in that Appendix violates unitarity by about a factor of six. If the value of the quantity ρ appearing there were approximately reduced to correct this violation, we would again find values for $F_1{}^{S\prime}(0)$ and $F_2{}^S(0)$ within the range of (4.11).

Our conclusion is then that the structure of the integral equations obtained by substituting (4.6) in (2.5) is such that the nucleon-pair intermediate state is quite unimportant and that a quantitative limit on the contribution of this state to $F_1{}'^S(0)$ and $F_2{}^S(0)$ is provided by the simple iteration scheme leading to Eq. (4.11).

V. CONCLUDING REMARKS

We have made quantitative estimates of all of the low-mass configurations which would be intuitively expected to dominate the low-momentum-transfer characteristics of nucleon structure. The theoretical understanding we have obtained is disappointingly small. Let us summarize the results of our investigation.

First, with regard to isotopic vector quantities, we had expected the two-pion intermediate state to provide the bulk of the contributions. This is probably true but it is not easy to say how true, since there is no way at present to make a believable calculation. It had been argued in C that, in fact, perturbation theory could be relied upon for a quantitative estimate. We have shown, however, by a rigorous argument based on unitarity, that once the mass σ of the two-pion intermediate state exceeds $2m$, the perturbation theory must be wrong. The rescattering corrections, together with the very likely appearance of a pion form factor, must reduce the contribution of the region $\sigma>2m$ to a very minor one. Now we cannot conclude from our argument that because perturbation theory is bad for $\sigma>2m$ it is necessarily bad in the unphysical region $\sigma<2m$ to which our unitarity argument does not apply. It seems very unreasonable however, to expect that the rescattering effects are suddenly unimportant for $\sigma<2m$.

The practical implication of these remarks is the following: The perturbation-theoretic value for the vector magnetic moment is $1.67e/2m$ which is quite close to the experimental value, $1.86e/2m$. Unfortunately, half of the value 1.67 comes from the region $\sigma>2m$ and must be almost totally discarded. We have evaluated the rescattering corrections using the method proposed in C, which involves a probably unwarranted analytic continuation of pion-nucleon scattering amplitudes; it is, however, the only known way of making an estimate. Aside from the questionable continuation procedure, we make no approximations beyond the familiar one of including only the (3,3) pion-nucleon scattering amplitude. We find then an additional contribution to the magnetic moment of about the right order of magnitude, namely about $1.03e/2m$, which together with the $0.87e/2m$ from perturbation theory makes for impressive agreement with experiment. Inclusion of the form factor for pions given in Fig. 1 reduces the total answer to $1.4e/2m$ which, considering that the $|2\pi\rangle$ state is one of an infinite number, would be quite satisfactory. The rescattering correction is, however, uncomfortably large (it must be judged independent of the pion form factor) and we have little confidence in it. It is not impossible that in fact even the sign of the correction is wrong, in which case the role of more massive states would become unpleasantly important. The quantitative situation for the magnetic moment is thus uncertain.

The mean square radius of the magnetization density is also affected by the rescattering corrections to perturbation theory and by the pion structure. Lowest order perturbation theory yields $\langle(r_2^2)_V\rangle=0.12/\mu^2$, a surprisingly small value, pointing again to the importance of large σ values in the dispersion integral. Inclusion of rescattering and the pion form factor causes this number to increase to $0.23/\mu^2$, a rather more reasonable value. The correspondingly experimental quantity is not directly obtainable. To the extent that the isotopic scalar contribution to the proton or neutron radius is negligible [which will be the case unless $\langle(r_2^2)_S\rangle\sim 20\langle(r_2^2)_V\rangle$], one may hope to find $\langle(r_2^2)_V\rangle$ from the electron-proton scattering experiments. Unfortunately, the magnetization density plays a minor role in the scattering until the momentum transfers are so large that the form factor F_2^P can no longer be characterized by a mean square radius alone. Extensive curve fitting in the large-momentum-transfer region has led Hofstadter and his collaborators to a value $\langle(r_2^2)_p\rangle=0.32/\mu^2$; in view of the way this number has been deduced it cannot be regarded as being in conflict with our theoretical number. *Unless one knows the actual functional form of the magnetization form factor, no amount of curve fitting at large momentum transfers can yield a meaningful value for the initial slope.*

The situation with respect to the charge density mean square radius is much more distressing. The prediction of lowest order perturbation theory is that $\langle(r_1^2)_V\rangle=0.24/\mu^2$; if we accept the empirical fact that for a neutron $\langle(r_1^2)_n\rangle\approx 0$ [which implies $\langle(r_1^2)_V\rangle=\langle(r_1^2)_S\rangle$] this leads to $\langle(r_1^2)_p\rangle=0.24/\mu^2$. The proton charge radius is in principle subject to direct measurement with electron scattering experiments at very low momentum transfer. What are needed are accurate absolute cross-section measurements in a region where the cross section is varying rapidly for uninteresting (i.e., ordinary Coulomb scattering) reasons and one is looking for small superimposed variations. The current interpretation of these difficult measurements by Hofstadter leads to a mean square charge radius of the proton of $0.32/\mu^2$. If this is, in fact, true, the theory may be in a very difficult position. The same rescattering effects which were helpful in connection with the magnetic moment have a devastating effect here. In the first place, the perturbation contribution from the region $\sigma>2m$ must be discarded; this amounts to about one-fourth of the above-mentioned value $0.24/\mu^2$. Secondly, the rescattering and pion form factor corrections in the region $\sigma<2m$ reduce the value further to $0.07/\mu^2$. Thus if the two-pion state is assumed to be the principal contributor to the isotopic vector charge radius, we appear to have a sharp disagreement with experiment. If we accept the small value of the neutron charge radius, we cannot rely on the isotopic scalar contribution to raise $0.07/\mu^2$ to $0.32/\mu^2$. It is obviously of crucial importance to obtain an accurate determination of the proton charge radius.

One may, of course, argue that the charge density form factor receives contributions from very high mass values and that we should not be too distressed by our inability to get a large radius from the two-pion configuration. It was, in fact, a certain fear of these high masses which led to our original forms for the dispersion relation for F_1^S, F_1^V. One argument supporting the importance of high mass values may be found by evaluating the integral over $\mathrm{Im}F_1^V(-\sigma^2)/\sigma^2$, which, if it exists, should equal $e/2$. Using our approximate rescattering and pion form factor we find instead approximately 0. As we have stated previously, we have very little confidence in our method of calculating the rescattering effects; it is not out of the question that the charge radius shows an extraordinary sensitivity to the pion-nucleon continuation method not shared by the magnetization density and that the large effects noted could be easily upset by changing the numbers slightly. Our inability to carry out a reliable calculation of the rescattering makes it difficult to draw any sharp conclusions.

In order to try to get some feeling for the possible contributions of higher mass configurations, we have studied several particular ones. The most important (in that it is the least massive) isotopic scalar contribution should come from the three-pion state. We were unable to even estimate this. The configuration of two K mesons was treated in perturbation theory; barring incredible accidents it is hard to see how this state can be of much importance. The nucleon-antinucleon pair states which have long played a rather enigmatic role in the nucleon structure problem have also been shown to make very small contributions to the moments and radii. Which, if any, of the high-mass states (i.e., $>2\pi$, 3π) are important we cannot say.

Of all of the quantities we have attempted to calculate, only the vector magnetic moment and magnetization density mean square radius appear to be reasonable. If our estimate of the two-pion state is reliable, we are unable to explain a proton charge radius any larger than about one-fifth the presently alleged value without an appeal to high-mass (and needless to say incalculable) configurations. If we somehow got a large vector charge radius we would still face the old dilemma of finding a sufficiently large isotopic scalar charge radius to explain the difference between neutron and proton. As has been emphasized by Yennie,[25] one then must also insure that this prolific source of charge radius should not yield too large a scalar magnetic moment.

In conclusion, there remain at least the following four alternatives: (1) The experiments are wrong and the proton charge radius is very small (this would obviously be the nicest solution for theoreticians). (2) The two-pion state is grossly mistreated in our theory so that one gets a large vector charge radius, and necessary isotopic scalar quantities ultimately appear,

[25] D. Yennie (private communication).

presumably from the three-pion state. (3) States of uncomputable complexity are important. (4) Our whole dispersion approach is wrong. The last possibility would be catastrophic.

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APPENDIX A. LADDER APPROXIMATION TO THE NUCLEON PAIR CONTRIBUTIONS TO THE ELECTROMAGNETIC STRUCTURE

We noted in Sec. IV that if the nucleon-antinucleon scattering encountered there were treated in perturbation theory, one would obtain a kind of ladder approximation to the vertex operator. An approach similar in spirit was made some time ago by Edwards.[10] His techniques were quite different and his results rather more ambiguous.

We proceed from Eq. (4.2) for the absorptive part of $A_\mu{}^J(N\bar{N})$ which for convenience we reproduce here:

$$A_\mu{}^J(N\bar{N})=-\pi\left(\frac{p_0}{m}\right)^{\frac{1}{2}}\sum\int\frac{d^3n\,d^3\bar{n}}{(2\pi)^3}\bar{v}(\bar{p})\times\langle 0|j_\mu|n\bar{n}\rangle\langle n\bar{n}|f|p\rangle\delta(n+\bar{n}-p-\bar{p}).\quad(\mathrm{A}.1)$$

The second matrix element is to be calculated in lowest order perturbation theory (with $f=-ig\gamma_5\tau_j\phi_j\psi+\delta m\psi$); we find

$$\left(\frac{n_0\bar{n}_0p_0}{m^3}\right)^{\frac{1}{2}}\bar{v}(p)\langle n\bar{n},\,\mathrm{out}|f|p\rangle=-g^2\frac{\bar{v}(\bar{p})\tau_j\gamma_5v(\bar{n})\bar{u}(n)\gamma_5\tau_ju(p)}{(n-p)^2+\mu^2-i\epsilon}+g^2\frac{\bar{u}(n)\tau_j\gamma_5\bar{v}(\bar{n})v(\bar{p})\tau_j\gamma_5u(p)}{(n+\bar{n})^2+\mu^2-i\epsilon}.\quad(\mathrm{A}.2)$$

It is convenient to use $\langle 0|j_\mu|n\bar{n}\rangle$ in the following form (Eq. 1.3):

$$(n_0\bar{n}_0/m^2)^{\frac{1}{2}}\langle 0|j_\mu|n\bar{n},\,\mathrm{out}\rangle=-\bar{v}(\bar{n})\{G_1{}^*[(n+\bar{n})^2]i\gamma_\mu+G_2{}^*[(n+\bar{n})^2](\bar{n}-n)_\mu\}u(n).\quad(\mathrm{A}.3)$$

When (A.3) and (A.4) are substituted into (A.1) and the spin sums carried out, the second term in (A.2) does not contribute. We find, in fact,

$$A_\mu{}^J(N\bar{N})=-\frac{\pi g^2}{4(2\pi)^3}\int\frac{d^3n\,d^3\bar{n}}{n_0\bar{n}_0}\frac{\delta(n+\bar{n}-p-\bar{p})}{(n-p)^2+\mu^2}\bar{v}(p)\tau_j\gamma_5\times(-i\gamma\cdot\bar{n}-m)[i\gamma_\mu\,\mathrm{Re}G_1+(\bar{n}-n)_\mu\,\mathrm{Re}G_2]\times(-i\gamma\cdot n+m)\tau_j\gamma_5u(p).\quad(\mathrm{A}.4)$$

We have, as usual, taken half the sum over "in" and "out" states and this accounts for the appearance of $\mathrm{Re}G_1$, $\mathrm{Re}G_2$. In the present approximation, the nucleon-antinucleon amplitude is real. [Compare (4.3) and the discussion following (4.4).] Recalling that the G's have the structure $G^S+\tau_3G^V$, we see that the effect of the τ_j's in (4.4) is to convert G into $\tilde{G}=3G^S-\tau_3G^V$. Finally we rewrite (A.4) in the form

$$A_\mu{}^J(N\bar{N})=\frac{-\pi g^2}{(2\pi)^3}\int d^4n\int d^4\bar{n}\int d^4k\,\frac{\bar{v}(\bar{p})i\gamma\cdot k[i\gamma_\mu\,\mathrm{Re}\tilde{G}_1-(\bar{n}-n)_\mu\,\mathrm{Re}\tilde{G}_2]i\gamma\cdot ku(p)}{k^2+\mu^2}\times\delta((p-k)^2+m^2)\delta((k+\bar{p})^2+m^2)\delta(n-p+k)\delta(\bar{n}-k-\bar{p})\quad(\mathrm{A}.5)$$

$$=-\frac{\pi}{2}\frac{g^2}{(2\pi)^3}\int d^4k\,\bar{v}(\bar{p})i\gamma\cdot k\frac{[i\gamma_\mu\,\mathrm{Re}\tilde{G}_1+(p-\bar{p}-2k)_\mu\,\mathrm{Re}\tilde{G}_2]i\gamma\cdot ku(p)}{k^2+\mu^2}\times\delta(k\cdot\Delta)\delta(k^2-P\cdot k),\quad(\mathrm{A}.6)$$

where we have introduced $\Delta=p+\bar{p}$, $P=p-\bar{p}$; note that $P^2=-4m^2-\Delta^2$.

The integrations in (A.6) are most easily carried out in the rest system of $p+\bar{p}$, namely $\Delta=0$. Comparison with the standard form (A.3) yields

$$\mathrm{Im}G_1(\Delta^2)=\frac{g^2}{4\pi}\frac{\theta(-4m^2-\Delta^2)}{8P^3(-\Delta^2)^{\frac{1}{2}}}\left\{P^4-2\mu^2P^2+2\mu^4\ln\frac{P^2+\mu^2}{\mu^2}\right\}\mathrm{Re}\tilde{G}_1(\Delta^2),\quad(\mathrm{A}.7)$$

$$\mathrm{Im}G_2(\Delta^2)=\frac{g^2}{4\pi}\frac{\theta(-4m^2-\Delta^2)}{2P(-\Delta^2)^{\frac{1}{2}}}\left[m\left\{-\frac{1}{2}+\frac{3\mu^2}{P^2}-\left(\frac{3\mu^4}{P^4}+\frac{\mu^2}{P^2}\right)\ln\frac{P^2+\mu^2}{\mu^2}\right\}\mathrm{Re}\tilde{G}_1(\Delta^2)+\left\{\mu^2-\left(\frac{\mu^2}{2}+\frac{\mu^4}{P_2}\right)\ln\frac{P^2+\mu^2}{\mu^2}\right\}\mathrm{Re}\tilde{G}_2(\Delta^2)\right].\quad(\mathrm{A}.8)$$

We shall be particularly interested in the scalar part of these equations. So long as states with 3, 5, $\cdots$ pions are neglected, the only source of isotopic scalar contributions is the pair state under discussion and the equations are entirely uncoupled from the two-pion state. Although it is not at all necessary, for simplicity we shall set the pion mass equal to zero. Experience with perturbation theory shows that this does not cause appreciable error. In this limit we have

$$\mathrm{Im}G_1{}^S(\Delta^2)=\frac{3}{8}\left(\frac{g^2}{4\pi}\right)\left(\frac{-4m^2-\Delta^2}{-\Delta^2}\right)^{\frac{1}{2}}\theta(-4m^2-\Delta^2)\times\mathrm{Re}G_1{}^S(\Delta^2), \quad (\mathrm{A.9})$$

$$\mathrm{Im}G_2{}^S(\Delta^2)=-\frac{3}{4}\left(\frac{g^2}{4\pi}\right)\frac{M\theta(-4m^2-\Delta^2)}{[(-\Delta^2-4m^2)(-\Delta^2)]^{\frac{1}{2}}}\times\mathrm{Re}G_1{}^S(\Delta^2) \quad (\mathrm{A.10})$$

$$=-2m\frac{\mathrm{Im}G_1{}^S(\Delta^2)}{(-\Delta^2-4m^2)}. \quad (\mathrm{A.11})$$

The mathematical problem posed by the condition (A.9) together with the dispersion relation

$$G_1{}^S(\Delta^2)=G_1{}^S(0)-\frac{\Delta^2}{\pi}\int_{4m^2}^{\infty}d\sigma^2\frac{\mathrm{Im}G_1{}^S(-\sigma^2)}{\sigma^2(\sigma^2+\Delta^2-i\epsilon)}, \quad (\mathrm{A.12})$$

is exactly the same as the one met earlier in connection with Eqs. (3.39) and (3.41), and the solution which agrees with perturbation theory in the limit of $g^2\rightarrow 0$ is

$$G_1{}^S(\Delta^2)=G_1{}^S(0)\exp\left\{-\frac{\Delta^2}{\pi}\int_{4m^2}^{\infty}d\sigma^2\frac{1}{\sigma^2(\sigma^2+\Delta^2-i\epsilon)}\times\tan^{-1}\left[\frac{3g^2}{32\pi}\left(\frac{\sigma^2-4m^2}{\sigma^2}\right)^{\frac{1}{2}}\right]\right\}. \quad (\mathrm{A.13})$$

The solution for $G_2{}^S(\Delta^2)$ is obtained immediately by substituting (A.11) into the dispersion relation for $G_2{}^S$:

$$G_2{}^S(\Delta^2)=\frac{1}{\pi}\int_{4m^2}^{\infty}d\sigma^2\frac{\mathrm{Im}G_2{}^S(-\sigma^2)}{\sigma^2+\Delta^2}$$

$$=-\frac{2m}{\pi}\int_{4m^2}^{\infty}d\sigma^2\frac{\mathrm{Im}G_1{}^S(-\sigma^2)}{(\sigma^2-4m^2)(\sigma^2+\Delta^2)}. \quad (\mathrm{A.14})$$

Making a partial-fraction decomposition, we find immediately

$$G_2{}^S(\Delta^2)=\frac{2m}{\Delta^2+4m^2}[G_1{}^S(\Delta^2)-G_1{}^S(-4m^2)]. \quad (\mathrm{A.15})$$

It is worth noting some of the easily obtained properties of these solutions. Recalling that by definition $G_2(0)$ is the scalar anomalous moment, μ_s, and that $G_1{}^S(0)=\frac{1}{2}e+2m\mu_s$, we find from (A.15)

$$\mu_s=\frac{1}{2}\left(\frac{1-\rho}{\rho}\right)\frac{e}{2m}, \quad (\mathrm{A.16})$$

where

$$\rho=\frac{G_1{}^S(-4m^2)}{G_1{}^S(0)}=\exp\left\{\frac{2}{\pi}\int_0^{3g^2/32\pi}dy\frac{\tan^{-1}y}{y}\right\}, \quad (\mathrm{A.17})$$

$$\simeq\exp\left[\frac{2}{\pi}\left(\frac{\pi}{2}\ln\lambda+\frac{1}{\lambda}-\frac{1}{3^2\lambda^3}+\frac{1}{5^2\lambda^5}-\cdots\right)\right], \quad (\mathrm{A.18})$$

and $\lambda=3g^2/32\pi\approx 5.6$. For this value of λ, $\rho=6.28$, and $\mu_s\simeq -0.42e/2m$. This value is about three times larger than the upper limit given in Eq. (4.11), a first indication of the inadequacy of our ladder approximation. The derivatives of $G_1{}^S$ and $G_2{}^S$ at $\Delta^2=0$ are also easily found:

$$G_1{}^{S\prime}(0)=-\frac{G_1{}^S(0)}{4\pi m^2}\left(\frac{\lambda^2+1}{\lambda^2}\tan^{-1}\lambda-\frac{1}{\lambda}\right),$$

$$\approx-\frac{e}{16m^2\rho}\left(1-\frac{2}{\pi\lambda}+\lambda^{-2}-\cdots\right) \quad (\mathrm{A.19})$$

$$=-0.010\frac{e}{m^2}.$$

$$G_2{}^{S\prime}(0)=-\frac{1}{4m^2}\mu_s+\frac{1}{2m}G_1{}^{S\prime}(0)=+\frac{0.095}{m^2}\frac{e}{2m},$$

$$F_1{}^{S\prime}(0)=G_1{}^{S\prime}-2mG_2{}^{S\prime}=\frac{1}{2m}\mu_s=-\frac{0.105e}{m^2}. \quad (\mathrm{A.20})$$

This value of $F_1{}^{S\prime}(0)$ is about nine times larger than the upper limit given in (4.11), again showing the poorness of the approximation.

In spite of the fact that the ladder model is not very good in any absolute sense, it does represent a considerable improvement over perturbation theory: The magnetic moment is reduced by a factor of about four, $G_1{}^{S\prime}(0)$ by about fifteen, and $G_2{}^{S\prime}(0)$ by about three. The asymptotic forms ($\Delta^2\rightarrow+\infty$) of (A.13) and (A.15) are quite different from perturbation theory for which $G_1\rightarrow\ln\Delta^2$ and $G_2\rightarrow\ln\Delta^2/\Delta^2$. In the present treatment we have

$$G_1{}^S(\Delta^2)\rightarrow G_1{}^S(0)\left(\frac{\Delta^2}{4m^2}\right)^{-(1/\pi)\arctan\lambda}$$

$$=G_1{}^S(0)\left(\frac{\Delta^2}{4m^2}\right)^{-0.44},$$

$$G_2{}^S(\Delta^2)\rightarrow-\frac{2m}{\Delta^2}G_1{}^S(-4m^2) \quad (\mathrm{A.21})$$

$$=-\frac{2m\rho G_1{}^S(0)}{\Delta^2}\simeq\frac{-G_1{}^S(0)}{2m}6.28\left(\frac{4m^2}{\Delta^2}\right).$$

The general form of our solution is similar to that found by Edwards[10] who studied certain aspects of the same problem. The two approaches are not directly comparable; he typically obtained asymptotic behaviors like $(\Delta^2)^{-\lambda/\pi}$ so that the results coincide for small λ. The solutions (A.13) and (A.15) cannot be expanded in powers of λ unless $\lambda<1$ which is not satisfied experimentally.

APPENDIX B. MESON-NUCLEON VERTEX

The meson nucleon vertex which describes the scattering of a nucleon by an external meson field occurs in a variety of problems such as π-μ decay and π^0 decay. The mathematical treatment of it follows so closely what has been developed in this paper that we felt it worthwhile to include the discussion here.

The quantity of interest, I, is defined according to Eq. (1.12) as

$$\begin{aligned} I&=(p_0'p_0/m^2)^{\frac{1}{2}}\langle p'|J_i|p\rangle \\ &=-i[\mu^2+(p-p')^2]\Delta_{Fc}[(p-p')^2]\bar{u}(p')\Gamma_5(p',p)u(p) \\ &=-ig\bar{u}(p')\tau_i\gamma_5 u(p)K((p-p')^2), \quad \text{(B.1)}\end{aligned}$$

where we imagine that $K(-\mu^2)=1$; the renormalized meson propagation function Δ_{Fc} is normalized so that the product of the first two factors in (B.1) is unity in the limit $(p-p')^2\to\mu^2$. Evidently we have to do with one scalar function, $K(\Delta^2)$, whose structure we must study.

As in the corresponding electromagnetic vertex, it is rather more convenient to study a quantity J defined by

$$\begin{aligned} J&=(\bar{p}_0p_0/m^2)^{\frac{1}{2}}\langle 0|J_i|\bar{p}p,\,\text{in}\rangle \\ &=ig\bar{v}(\bar{p})\tau_i\gamma_5 u(p)K((p+\bar{p})^2); \quad \text{(B.2)}\end{aligned}$$

the analyticity assumption we customarily make enable us to relate the scalar functions involved in I and J in the simple fashion indicated. Making the standard reduction, dropping an equal-time commutator as usual, we have

$$\begin{aligned} J=-i\left(\frac{p_0}{m}\right)^{\frac{1}{2}}\int d^4x\,\bar{v}(\bar{p}) \\ \times\langle 0|[J_i(0),f(x)]\theta(-x_0)|p\rangle\exp(i\bar{p}\cdot x). \quad \text{(B.3)}\end{aligned}$$

The absorptive part, A_J, is given by

$$\begin{aligned} A_J=-\pi(p_0/m)^{\frac{1}{2}}\textstyle\sum_s \bar{v}(\bar{p})\langle 0|J_i|s\rangle \\ \times\langle s|f|p\rangle\delta(p_s-p-\bar{p}). \quad \text{(B.4)}\end{aligned}$$

The least massive intermediate state is that consisting of three pions, the neglect of which by this time scarcely needs comment. The first state to be considered is that involving a nucleon-antinucleon pair, and, as always, we take half the sum over "in" and "out" states.

We shall first treat this problem in the same way we did in Appendix A, namely by describing the matrix element $\langle N\bar{N}|f|p\rangle$ in lowest order perturbation theory. In fact, we use Eq. (A.2) for the matrix element. Substituting (A.2) and (B.2) into (B.4) (with $|s\rangle$ taken as $|\bar{N}N\rangle$) and carrying out the indicated operations, just as in (A.6), we find by comparison with (B.2)

$$\begin{aligned} \mathrm{Im}K(\Delta^2)=\frac{g^2}{16\pi}\frac{\theta(-\Delta^2-4m^2)}{\sqrt{-\Delta^2}} \\ \times\left[P-\frac{\mu^2}{P}\ln\frac{P^2+\mu^2}{\mu^2}-\frac{4\Delta^2P}{\Delta^2+\mu^2}\right]\mathrm{Re}K(\Delta^2), \quad \text{(B.5)}\end{aligned}$$

where $P^2=-\Delta^2-4m^2$. The first two terms in (B.5) come from the first term in (A.2) whereas the last comes from the second term. We have purposely separated these contributions, since the first two terms in (B.5) comprise what may be called the proper vertex part $\bar{v}(\bar{p})\Gamma_5(\bar{p},p)u(p)$, while the second comes from a nucleon bubble in a meson propagation function and represents the deviation of $(\Delta^2+\mu^2)\Delta_{Fc}(\Delta^2)$ from unity. As we'll see in a moment, keeping or dropping the modified propagation function has a very profound effect.

For simplicity we set the pion mass equal to zero. Then if we drop the last term in (B.5), we will have to do with the proper vertex part; and this quantity, which we shall call $\Gamma(\Delta^2)$, is directly comparable with the ladder approximation to G_1 in Appendix A. For Γ, we have then

$$\begin{aligned} \mathrm{Im}\Gamma(\Delta^2)=\frac{g^2}{16\pi}\left(\frac{-\Delta^2-4m^2}{-\Delta^2}\right)^{\frac{1}{2}} \\ \times\mathrm{Re}\Gamma(\Delta^2)\theta(-\Delta^2-4m^2). \quad \text{(B.6)}\end{aligned}$$

The structure of (B.6) is exactly that of (A.9); since the dispersion relation in this case is taken to be

$$\Gamma(\Delta^2)=1-\frac{\Delta^2+\mu^2}{\pi}\int_{4m^2}^{\infty}d\sigma^2\frac{\mathrm{Im}\Gamma(-\sigma^2)}{(\sigma^2-\mu^2)(\sigma^2+\Delta^2-i\epsilon)}, \quad \text{(B.7)}$$

the solution of the integral equation is

$$\begin{aligned} \Gamma(\Delta^2)=\exp\left\{\frac{-(\Delta^2+\mu^2)}{\pi}\int_{4m^2}^{\infty}d\sigma^2\frac{1}{(\sigma^2-\mu^2)(\sigma^2+\Delta^2-i\epsilon)}\right. \\ \left.\times\tan^{-1}\left[\frac{g^2}{16\pi}\left(\frac{\sigma^2-4m^2}{\sigma^2}\right)^{\frac{1}{2}}\right]\right\}, \quad \text{(B.8)}\end{aligned}$$

and it has been chosen to agree with perturbation theory in the limit of small g^2. The asymptotic form of $\Gamma(\Delta^2)$ is easily seen to be

$$\Gamma(\Delta^2)\to(4m^2/\Delta^2)^{(1/\pi)\arctan(g^2/16\pi)}. \quad \text{(B.9)}$$

If we keep both the proper vertex part and the propagation function modification, i.e., the whole of

(B.5), with $\mu^2=0$ we have

$$\mathrm{Im}K(\Delta^2)=-\frac{3g^2}{16\pi}\left(\frac{-\Delta^2-4m^2}{-\Delta^2}\right)^{\frac{1}{2}}\times\mathrm{Re}K(\Delta^2)\theta(-\Delta^2-4m^2), \quad (B.10)$$

and the solution corresponding to (B.8) is evidently

$$K(\Delta^2)=\exp\left\{\frac{\Delta^2+\mu^2}{\pi}\int_{4m^2}^{\infty}d\sigma^2\frac{1}{(\sigma^2-\mu^2)(\sigma^2+\Delta^2-i\epsilon)}\times\left[\tan^{-1}\left(\frac{3g^2}{16\pi}\right)\right]\left(\frac{\sigma^2-4m^2}{\sigma^2}\right)^{\frac{1}{2}}\right\}. \quad (B.11)$$

The asymptotic form of $K(\Delta^2)$ is easily seen to be

$$K(\Delta^2)\rightarrow(\Delta^2/m^2)^{(1/\pi)\arctan(3g^2/16\pi)}. \quad (B.12)$$

Thus we see that $\Gamma(\Delta^2)$ approaches zero for large Δ^2 whereas $K(\Delta^2)$ increases indefinitely in the present approximation.

This behavior does not contradict any general principles. Making reasonable assumptions about the propagation function Δ_{Fc} it has been shown by Lehmann *et al.*,[18] quite generally, that $\Gamma(\Delta^2)\rightarrow 0$ as $\Delta^2\rightarrow\infty$. On the other hand $[\Delta^2+\mu^2]\Delta_{Fc}((\Delta^2+\mu^2))$ in the limit of large Δ^2 approaches the renormalization constant $1/Z_3$ which is generally believed to be infinite, so that having K increase is not necessarily unreasonable. It is amusing to combine (B.11) and (B.8) to give a formula for Δ_{Fc} which sums nucleon bubbles with various numbers of ladders going across the bubbles. We find

$$\Delta_{Fc}(\Delta^2)=\frac{1}{\Delta^2+\mu^2}\exp\left\{\left(\frac{\Delta^2+\mu^2}{\pi}\right)\times\int_{4m^2}^{\infty}d\sigma^2\frac{1}{(\sigma^2-\mu^2)(\sigma^2+\Delta^2-i\epsilon)}\times\left[\left(\frac{\sigma^2-4m^2}{\sigma^2}\right)^{\frac{1}{2}}\tan^{-1}\left(\frac{3g^2}{16\pi}\right)+\left(\frac{\sigma^2-4m^2}{\sigma^2}\right)^{\frac{1}{2}}\tan^{-1}\left(\frac{g^2}{16\pi}\right)\right]\right\} \quad (B.13)$$

$$\rightarrow\frac{1}{\Delta^2}\times\left(\frac{\Delta^2}{4m^2}\right)^{\kappa}, \quad (B.14)$$

where

$$\kappa=\frac{1}{\pi}\left[\tan^{-1}\left(\frac{3g^2}{16\pi}\right)+\tan^{-1}\left(\frac{g^2}{16\pi}\right)\right]\simeq 1-\frac{16}{3\pi}\left(\frac{4\pi}{g^2}\right)+\cdots. \quad (B.15)$$

Thus $\Delta_{Fc}\rightarrow 0$ rather slowly for large g^2 in this approximation. The results of our ladder approximation to $\Gamma(\Delta^2)$ are quite similar to those of Edwards.[10]

We turn finally to a more accurate treatment of the problem. Instead of using lowest order perturbation theory to describe the nucleon-antinucleon scattering amplitude $\bar{v}(\bar{p})\langle N\bar{N}|f|p\rangle$, we shall characterize it by a complex phase shift. The crucial observation which makes this possible is that the matrix element $\langle 0|J_i|N\bar{N}\rangle$ is different from zero only if the nucleon-antinucleon pair are (in their rest system) in a state of angular momentum zero, odd parity, and isotopic spin unity. There is only one such state, namely ${}^1S_0{}^3$ where the superscript 3 designates the isotopic triplet. We write now, in the rest frame of the pair,

$$J=(p_0/m)ig\chi_{\bar{p}}{}^*\tau_i\chi_pK((p+\bar{p})^2), \quad (B.16)$$

$$\bar{v}(\bar{p})\langle N\bar{N}|f|p\rangle=-\frac{4\pi n_0}{m^2 n}\times\left(\frac{m^3}{n_0{}^3}\right)^{\frac{1}{2}}\frac{3\chi_{\bar{p}}{}^*\chi_{\bar{n}}\chi_n{}^*\chi_p-\chi_{\bar{p}}{}^*\boldsymbol{\tau}\chi_{\bar{n}}\cdot\chi_n{}^*\boldsymbol{\tau}\chi_p}{4}\beta_0, \quad (B.17)$$

where here

$$\beta_0=\sin\delta\exp(i\delta), \quad (B.18)$$

and δ is the complex phase shift for the ${}^1S_0{}^3$ nucleon-antinucleon state and is a function of wave number, $[-\frac{1}{4}(p+\bar{p})^2-m^2]^{\frac{1}{2}}$.

The calculation is now trivial and follows what should by now be well established patterns. Substituting (B.16) and (B.17) into (B.4), we find by referring to (B.16)

$$\mathrm{Im}K(\Delta^2)=\mathrm{Re}(K^*\beta_0), \quad (B.19)$$

from which we obtain

$$\mathrm{Im}K(\Delta^2)=\frac{\mathrm{Re}\beta_0}{1-\mathrm{Im}\beta_0}\mathrm{Re}K(\Delta^2). \quad (B.20)$$

This is of the standard form discussed in Sec. III in connection with the pion vertex. The phase angle φ defined there has exactly the same form in this case,

$$\tan\varphi(-\Delta^2)=\mathrm{Re}\beta_0/(1-\mathrm{Im}\beta_0). \quad (B.21)$$

If we fix φ by the requirement of $\varphi=0$ at zero wave-number ($-\frac{1}{4}\Delta^2=m^2$) and further assume that K has no zero, we have

$$K(\Delta^2)=\exp\left[-\frac{(\Delta^2+\mu^2)}{\pi}\times\int_{4m^2}^{\infty}d\sigma^2\frac{\varphi(\sigma^2)}{(\sigma^2-\mu^2)(\sigma^2+\Delta^2-i\epsilon)}\right] \quad (B.22)$$

in accordance with our normalization convention.

The asymptotic form of K depends on what happens to φ for large σ^2. If $\varphi \to 0$ at infinity, $K(\Delta^2) \to 1$; if φ approaches a positive constant, $K \to 0$. Within the framework of our model as expressed by (B.21), if there is any absorption whatsoever, φ is less than $\pi/2$. This may be seen by writing $\delta_0 = \xi + i\eta$ and noting that

$$\tan\varphi = \frac{e^{-2\eta}\sin 2\xi}{1+e^{-2\eta}\cos 2\xi}. \tag{B.23}$$

It may be argued that since in (B.4) we dropped reference to all states other than that involving a pair, we have no right to contemplate complex δ_0's, for it is just those states which lead to the complexity of the phase. It is our feeling, however, that there is sense to our procedure, since what we require for the validity of the approximation made is confined specifically to (B.4): The other terms may be small because of $\langle 0|J_i|s\rangle$ being small irrespective of the structure of the other factor. This point is also discussed in reference 11 where in addition the structure of K is examined for some simple models.

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Interaction Current in Strangeness-Violating Decays*

S. Okubo and R. E. Marshak, *University of Rochester, Rochester, New York*
E. C. G. Sudarshan, *Harvard University, Cambridge, Massachusetts*
W. B. Teutsch, *Tufts University, Medford, Massachusetts*

AND

S. Weinberg, *Columbia University, New York, New York*

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The consequences of the hypothesis that the current of strongly interacting particles contributing to strangeness-violating decays has the transformation properties of an isospinor are investigated. The six processes $K^+\to\pi^0+\mu^++\nu$, $K^-\to\pi^0+\mu^-+\bar{\nu}$, $K_1^0\to\pi^++\mu^-+\bar{\nu}$, $K_1^0\to\pi^-+\mu^++\nu$, $K_2^0\to\pi^++\mu^-+\bar{\nu}$, $K_2^0\to\pi^-+\mu^++\nu$ would then have the same rates, angular correlations, spectra, etc., and likewise for electron modes. This prediction is compared with available experimental data. The evidence from the nonlepton decays is briefly examined.

1. INTRODUCTION

Recent experimental and theoretical developments[1] lend some support to a rather specific form of the interaction responsible for weak decays which conserve strangeness. This interaction can be considered as the self coupling of a chiral current, the current itself being constructed additively from baryon and lepton parts. The structure of the lepton current is quite well determined (provided the leptons are assumed to have no strong couplings) but the same cannot be said of the baryon currents in view of the complications introduced by the strong interactions. Nevertheless the present experimental position in β decay is consistent with a chiral baryon current with the transformation properties of an isotopic vector.[2]

In the decays of strange particles, one has two general groups of decay modes according as whether there are leptons in the final state or not. For those cases in which there are no leptons, the expression of the weak interaction as the coupling of a strangeness-conserving current with the strangeness-nonconserving current is consistent with all experimental data. One notices that while in principle the transition matrix element depends on the interaction in a definite manner, lack of adequate methods of studying such a system of strongly interacting particles makes this information on the currents practically inaccessible. In particular cases quantitative estimates of the interaction can be made and the decay of the Λ hyperon is such a case.[3] But in general, one must look to the lepton decay modes for direct information on the strangeness-violating current of strongly interacting particles, since here the transition matrix element is simply expressed in terms of the currents $\mathcal{J}_B{}^\lambda$ [see Eq. (1)].

In the case of the strangeness-conserving decays, the current coupled to the leptons has transformation properties similar to the positive chiral part of the current to which pseudoscalar mesons are pseudovector coupled. Arguing from analogy, one is thus led to postulate that the strangeness-violating current coupled to leptons has isotopic spin transformation properties similar to the positive chiral part of the current to which pseudoscalar K mesons are pseudovector coupled. We shall not discuss here the consequences of the

* Some of these results were presented at the 1958 International Conference on High Energy Physics at CERN.

[1] E. C. G. Sudarshan and R. E. Marshak, Proceedings of the Padua-Venice Conference, September, 1957 (to be published); Phys. Rev. **109**, 1860 (1958). R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[2] See M. Gell-Mann (to be published).

[3] Okubo, Marshak, and Sudarshan (to be published).